[06-12-05-T12]

Prove MVT

■ MVT

If f is continuous on a closed interval $[a, b]$ and differentiable on its interior (a, b), then there is at least one number c in (a, b) where

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

or, equivalently, where

$$f(b) - f(a) = f'(c)(b-a)$$

Proof.
Suppose f is continuous on a closed interval $[a, b]$ and differentiable on its interior (a, b). Consider the function s defined by

$$s(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$$

Notice $s(a) = s(b) = 0$, s is continuous on $[a, b]$, and s is differentiable on (a, b). Differentiating,

$$s'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}.$$

If $\exists\, c \in (a, b) \ni s'(c) = 0$, the conclusion, $f'(c) = \frac{f(b)-f(a)}{b-a}$, follows immediately.

Since s is the diference of two continuous functions, s is continuous on $[a, b]$. The Min-Max Existence Thm guarantees that s attains both a maximum and minimum value on $[a, b]$.

Suppose $s(x)\,\text{max} = s(x)\,\text{min} = 0$. Then $s(x)$ is constant on $[a, b]$ and $s'(x) = 0,\ \forall\, x \in (a, b)$; that is, every $x \in (a, b)$ is the c we desire to find.

Alternatively, suppose that $s(x)\,\text{max} \neq 0$ or $s(x)\,\text{min} \neq 0$. Then s attains this extreme value at an interior point of $[a, b]$. Since s is differentiable everywhere in (a, b), no singular point exists on (a, b). So, by the Critical Point Thm, the extreme value must occur at a stationary point; that is at $c \in (a, b) \ni s'(c) = 0$.

In any case, there is a c in (a, b) for which $s'(c) = 0$, hence $f'(c) - \frac{f(b)-f(a)}{b-a} = 0$. Therefore, $\exists\, c \in (a, b) \ni f'(c) = \frac{f(b)-f(a)}{b-a}$.

□