[07-01-04-T12]

33. Establish **Simpson's rule**

$$\int_a^b f = \lim_{n\to\infty} S_n,$$

where $S_n = \frac{1}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 4y_{n-1} + y_n)h$, where n is even Moreover, show that for some $c \in [a, b]$

$$\int_a^b f - S_n = -\frac{(b-a)h^4 f''''(c)}{180}.$$

34. Establish **midpoint rule** $\int_a^b f = \lim_{n\to\infty} M_n$, where

$$M_n = h\sum_{k=1}^{n} f\left(a + \left(k - \frac{1}{2}\right)h\right)$$

35. Why is Simpson's rule a generally more effective method for numerical integration than the rectangular rule?

36. For a polynomial $P(x)$ of degree at most 3, show that Simpson's rule yields the exact value.

37. Use the Simpson and the trapezoidal rules with $n = 4$ and 8, respectively, to obtain approximations for $\int_0^1 f$ for the functions

(a) $f(x) = \frac{1}{x}$; (b) $f(x) = \frac{1}{1+x^2}$; $f(x) = e^{-x^2}$.

Compare the two results.

38. If $|f''(x)| < M$ in $[0, 1]$, show that the error in calculating $\int_0^1 f$ by Simpson's rule is less than $\frac{M}{2880n^4}$, where the interval $[0, 1]$ is divided into $2n$ equal parts.

5.5 The Relationship Between Integration and Differentiation

The preceding sections saw the development of the notion of integral as a limit of approximating sums that have no obvious relationship to the process of differentiation. In this section, we bring forth the intimate connection between the notions of differentiation and integration for a certain class of functions and show that in the case of continuous functions, the so-called Fundamental Theorem of Integral Calculus is true; namely, integration is the reverse process of differentiation. To begin our discussion, we associate with a function f defined on the interval $[a, b]$, a new function that has the role of an 'antiderivative'. More precisely,

Definition. If f is a function defined on $[a, b]$, a function F is called a **primitive** (or an**antiderivative**) of f on $[a, b]$, provided F is differentiable and $F'(x) = f(x)$ for all $x \in [a, b]$.

From: Elements of Real Analysis, Gaskill & Narayanaswami

Discussion. The essential point about antiderivatives is that they are not unique. However, the key additional fact regarding primitives is that two functions that are primitives for the same function must differ by a constant. □

Theorem 5.5.1. *Let F and G be primitives for f on $[a, b]$. Then $F - G$ is a constant on $[a, b]$.*

Proof. Let $H = F - G$ on $[a, b]$. The reader can check that H satisfies the hypothesis of the Mean Value Theorem on $[a, b]$. Suppose, for the sake of argument, that H is not constant of $[a, b]$. Then there exist $c, d \in [a, b]$ such that $c < d$ and $H(c) \neq H(d)$. By the Mean Value Theorem, we can find $x \in (a, b)$ such that

$$H'(x) = \frac{H(d) - H(c)}{d - c} \neq 0.$$

However, $H'(x) = 0$ for all $x \in [c, d]$, whence we have a contradiction. □

Discussion. The proof of this theorem is another of the many applications we have seen for the Mean Value Theorem. The reader should come to an understanding of why the Mean Value Theorem is the natural tool to apply in this context. An attempt to construct a more direct proof from the definition of derivative might aid in developing this appreciation.

The converse of this theorem is left as Exercise 1. The theorem and its converse imply that if a function has a primitive, then it has an infinite number of primitives. □

The next result that brings forth the relationship between integrals and primitives enables us to evaluate a Riemann Integral easily, provided we have the knowledge and access to a primitive of that function. However, not all functions possess primitives, and the theorem is applicable only if the primitive is already available in advance.

Theorem 5.5.2 (Fundamental Theorem of Calculus). *If f is integrable on $[a, b]$ and F is any primitive for f on $[a, b]$, then*

$$\int_a^b f = F(b) - F(a).$$

Proof. Let $P = \{a = x_0, x_1, \ldots, x_n = b\}$ be any partition of $[a, b]$. Consider the interval $[x_{i-1}, x_i]$. By the Mean Value Theorem (Theorem 4.3.3), we have that for some $c_i \in (x_{i-1}, x_i)$

$$F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1}) = f(c_i)d_i,$$

where $d_i = x_i - x_{i-1}$. Let Q be the intermediate partition of P consisting of the c_i 's. It is immediate that

$$\begin{aligned} F(b) - F(a) &= \sum [F(x_i) - F(x_{i-1})] \\ &= \sum f(c_i)d_i = R(f, P, Q), \end{aligned}$$

a Riemann sum. Thus, for every partition, P, regardless of norm, there is an intermediate partition, Q, such that $R(f,P,Q) = F(b) - F(a)$. Hence, we conclude

$$\int_a^b f = F(b) - F(a). \qquad \square$$

Corollary. *If the derivative, f', of f is integrable on $[a,b]$, then*

$$\int_a^b f' = f(b) - f(a).$$

Discussion. Theorem 5.5.2 is known as the Fundamental Theorem of Calculus. Its purpose is to make the process of integration easier. Specifically, to obtain the integral of f on $[a,b]$, find *any* primitive, F, of f; evaluate this primitive at a and b, and finally, compute $F(b) - F(a)$. We emphasize that any primitive will do. It is clear that when a primitive is known, the process described above is much simpler than trying to use the methods suggested in the previous section. However, as the reader well knows from previous studies, the work involved in finding a primitive for a given function can be substantial, and as we have already remarked, a simple primitive may not exist. It is clear then that we should try to elucidate the conditions that will guarantee the existence of a primitive, F, although mere existence will not guarantee that we can produce a primitive in any kind of a usable form. Moreover, simply because a function has a primitive on a given closed interval does not ensure that it is integrable there (see Exercise 8).

Let us turn now to the proof of the theorem. As with the previous theorem, it is an application of the Mean Value Theorem, and a beautifully simple one at that. The key line is the proof is

$$F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1}) = f(c_i)(x_i - x_{i-1}).$$

The essential fact that must be understood is that if we sum over i, the left-hand side sums to $F(b) - F(a)$, while the right-hand side is a Riemann sum. Once this is realized, letting the norm of the partition go to 0 completes the proof.

The Mean Value Theorem also provides possible insight about the original thinking that led to the Fundamental Theorem. Consider the problem of finding the area under a curve, as shown in Figure 5.5.1. Suppose the function, F, evaluated at c yields the area bounded by the graph of f, the x-axis, the lines $x = a$ and $x = c$ for any $c \in [a,b]$. The portion of the figure bounded by $x = c$ and $x = c + \Delta x$ has an area is given by

$$F(c + \Delta x) - F(c) = f(c_0) \cdot \Delta x,$$

where $c_0 \in [c, c + \Delta x]$. One could supply a variety of arguments for this position, but the initial one was probably that it was 'intuitively obvious'. In any case, if one supposes that F is differentiable, then division by Δx yields

$$\frac{F(c + \Delta x) - f(c)}{\Delta x} = f(c_0).$$

Since the left-hand side contains a quantity that in the limit becomes the derivative of F, one concludes that f must be the derivative of F.

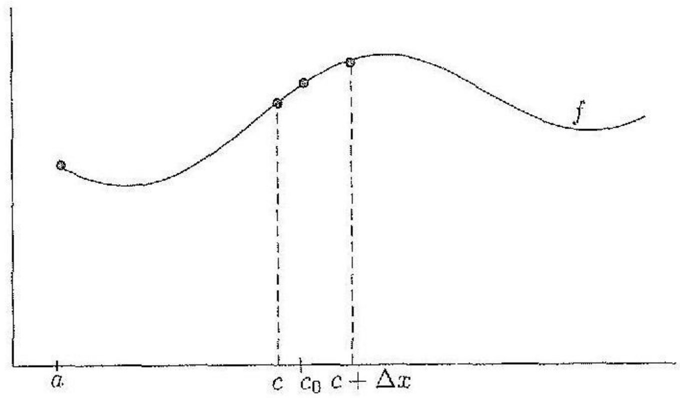

Figure 5.5.1 An increment of the area under the graph of f; the numerical value is given by $F(c+\Delta x) - F(c)$.

The Fundamental Theorem also gives rise to the more standard notation for integrals. If we set $\Delta x_i = x_i - x_{i-1}$, then

$$\int_a^b f = \lim_{\|P\|\to 0} R(f,P,Q) = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i.$$

In the limit, $\Delta x \to dx$, which may be thought of as an infinitesmally small number. This forces $f(c_i) \to f(x)$, where $x \in [c_i, c_i + dx]$. Thus we have the notational form

$$\int_a^b f(x)dx = \int_a^b f. \qquad \square$$

Our aim now is to find a condition on an **integrand** (function to be integrated) that will guarantee the existence of a primitive.

Theorem 5.5.3. *Let f be integrable on $[a,b]$, and define*

$$F(x) = \int_a^x f, \ x \in [a,b],$$

then F is a continuous function with domain $[a,b]$. Further, if $c \in (a,b)$ and f is continuous at c, then F is differentiable at c and $F'(c) = f(c)$.

Proof. The fact that $F(x)$ is defined for each $x \in [a,b]$ follows from Theorem 5.4.7. That F is a function follows from the fact that the limit of a Riemann sum is unique. The continuity of F is left as Exercise 2.

To establish differentiability, consider the quantity $\frac{F(c+h)-F(c)}{h}$, where h is small enough to satisfy $c+h \in (a,b)$. From the definition of F, we have

$$\frac{F(c+h)-F(c)}{h} = \frac{1}{h}\int_c^{c+h} f.$$