[07-02-21-L-11]

Property 2 was given and used to solve exponential equations in Section 11.1.

1 The following examples show how these properties are used to solve equations. The first examples illustrate a general method, using Property 3, for solving exponential equations.

Example 1 Solve the equation $3^m = 12$.

Use Property 3 to write

$$\log 3^m = \log 12.$$

By the power rule for logarithms, $\log 3^m = m \log 3$, or

$$m \log 3 = \log 12$$
$$m = \frac{\log 12}{\log 3}.$$

This is the exact solution. To get a decimal approximation for the solution, use a logarithm table or a calculator. From the logarithm table at the back of the book,

$$m = \frac{1.0792}{.4771}$$
$$m \approx 2.262. \qquad \text{Divide 1.0792 by .4771}$$

The solution set is $\{(\log 12)/\log 3\}$ (exactly) or $\{2.262\}$ (approximately). ✣

Example 2 Solve $\left(\frac{1}{16}\right)^{2x+3} = 8^{(5x+1)/3}$.

Since both $1/16$ $(= 2^{-4})$ and 8 $(= 2^3)$ are powers of 2, take base 2 logarithms of both sides.

$$\log_2 \left(\frac{1}{16}\right)^{2x+3} = \log_2 8^{(5x+1)/3} \qquad \text{Property 3}$$

$$(2x + 3) \log_2 \left(\frac{1}{16}\right) = \left(\frac{5x + 1}{3}\right) \log_2 8 \qquad \text{Power rule for logarithms}$$

Since $\log_2 \frac{1}{16} = -4$ and $\log_2 8 = 3$,

$$(2x + 3)(-4) = \left(\frac{5x + 1}{3}\right)(3)$$
$$-8x - 12 = 5x + 1$$
$$x = -1.$$

A check will show that $\{-1\}$ is the solution set of the original equation. This equation also could be solved using base 10 logarithms as in Example 1. Another method would be to write each side as a power of 2 and then use Property 2 as in Section 11.1. ✣

2 The next three examples illustrate ways to solve equations with logarithms. The properties of logarithms from Section 11.3 are useful here, as is using the definition of logarithm to change to exponential form.

Example 3 Solve the equation $\log_2 (x + 5)^3 = 4$.

Using the definition of logarithm gives $(x + 5)^3 = 2^4$, or

$$\begin{aligned} (x + 5)^3 &= 16 \\ x + 5 &= \sqrt[3]{16} \\ x &= -5 + \sqrt[3]{16}. \end{aligned}$$

Verify that the solution satisfies the equation, so that the solution set is $\{-5 + \sqrt[3]{16}\}$. ✥

Recall that the domain of $y = \log_b x$ is $\{x|x > 0\}$. For this reason,

it is always necessary to check that the solution of an equation with logarithms is in the domain.

Example 4 Solve $\log_2 (x + 1) - \log_2 x = \log_2 8$.

Use the quotient rule for logarithms, then Property 4 from the beginning of this section.

$$\log_2 (x + 1) - \log_2 x = \log_2 \frac{x + 1}{x} \qquad \text{Quotient rule for logarithms}$$

The original equation becomes

$$\begin{aligned} \log_2 \frac{x + 1}{x} &= \log_2 8 \\ \frac{x + 1}{x} &= 8 && \text{Property 4} \\ 8x &= x + 1 \\ x &= \frac{1}{7}. \end{aligned}$$

Check this by substitution in the given equation. Here both $x + 1$ and x must be positive. If $x = 1/7$, this condition is satisfied, making the solution set $\{1/7\}$. ✥

Example 5 Solve $\log x + \log \frac{3x}{2} = 5$.

Use the product rule for logarithms to write

$$\log x + \log \frac{3x}{2} = \log \frac{3x^2}{2}.$$

Also, the log notation indicates that the base is 10, giving the equation

$$\log_{10}\left(\frac{3x^2}{2}\right) = 5.$$

Using the definition of logarithm to switch to exponential form gives

$$\begin{aligned} \frac{3x^2}{2} &= 10^5 \\ \frac{3x^2}{2} &= 100{,}000 \\ 3x^2 &= 200{,}000 \\ x^2 &= \frac{200{,}000}{3} \\ x &\approx 258.2. \end{aligned}$$

The negative square root, -258.2, cannot be used as a solution, since x must be positive. The solution set is $\{258.2\}$ (approximately). ✣

3 The final two examples show applications of exponential equations.

Example 6 When interest is compounded (interest is paid on interest), P dollars deposited at a rate of interest i compounded annually for t years becomes

$$A = P(1 + i)^t.$$

For example, \$5000 deposited at 6% for 7 years becomes

$$A = 5000(1 + .06)^7.$$

Logarithms can be used to find an approximate value for this amount.

$$\begin{aligned} \log 5000(1 + .06)^7 &= \log 5000 + \log (1.06)^7 \\ &= \log 5000 + 7 \log 1.06 \\ &= 3.6990 + 7(0.0253) \\ &= 3.6990 + 0.1771 \\ &= 3.8761 \end{aligned}$$

Find the antilogarithm from the table; \$5000 would become about \$7520 in 6 years at 6% interest. Using a calculator to evaluate $(1.06)^7$ gives a result of \$7518.15. ✣

Example 7 Suppose that over the years the average annual rate of inflation is 7%. How long would it take for the average level of prices to double?

Use the formula from Example 6,

$$A = P(1 + i)^t.$$

Answers to exercise that follows

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1. {3/2} **3.** {−4.75} **5.** {.16} **7.** {−3.32} **9.** {−.29} **11.** {−1.43, 1.43} **13.** {4} **15.** {2} **17.** ∅ **19.** {2} **21.** {8} **23.** {3.317} **25.** {.117} **27.** Exact answer (using a calculator) is \$11,260.95 **29.** \$613.91 **31.** (a) 23.4 years (b) 14.2 years (c) 11.9 years (d) 9.0 years **33.** {1/2} **35.** {3.71} **37.** {1/2, 1} **39.** {152} **41.** {1/2} **43.** {10/3} **45.** 2/3 **47.** −1/2

11.5 Exercises

A calculator with a log key can be used to work the exercises in this section.

Solve the following equations. Round solutions to the nearest hundredth. See Examples 1 and 2.

1. $25^x = 125$ **2.** $16^x = 64$ **3.** $2^{-x} = 27$ **4.** $8^{-p} = 12$

5. $6^{y+1} = 8$ **6.** $2^{-3+y} = 4.5$ **7.** $\left(\frac{1}{2}\right)^x = 10$ **8.** $3^{y+1} = 2$

9. $5^{1-n} = 8$ **10.** $7^{2x+1} = 3$ **11.** $4^{x^2} = 17$ **12.** $3^{-x^2} = .09$

Solve the following equations. See Examples 3–5.

13. $\log (x + 2) = \log (3x - 6)$

14. $\log x = \log (1 - x)$

15. $\log_5 (3x + 2) - \log_5 x = \log_5 4$

16. $\log_2 (x + 5) - \log_2 (x - 1) = \log_2 3$

17. $\log 4x = \log 2 + \log (x - 3)$

18. $\log (-x) + \log 3 = \log (2x - 15)$

19. $\log x + \log (3x - 5) = \log 2$

20. $\log (6x - 7) + \log x = \log 5$

21. $\log_2 x = 3$

22. $\log_x 10 = 3$

23. $\log_y 11 = 2$

24. $\log_m 4 = \frac{3}{2}$

25. $\log_a 5 = -\frac{3}{4}$

26. $2 + \log x = 0$

Solve the following problems. See Examples 6 and 7.

27. Find the amount of money in an account after 12 years if \$5000 is deposited at 7% compounded annually.

28. How much money will be in an account at the end of 8 years if \$4500 is deposited at 6% compounded annually?

29. How much money must be deposited today to amount to \$1000 in 10 years at 5% compounded annually?

30. How much money must be deposited today to become \$1850 in 40 years at 6.5% compounded annually?

31. How long would it take for the average price level to double if the average rate of inflation annually is: (a) 3%; (b) 5%; (c) 6%; (d) 8%. (Check your answers by using the *rule of 70:* The time for prices to double is given by $70/x$, where x is the percent of annual inflation.)

32. A machine purchased for business use *depreciates,* or wears out, over a period of years. The value of the machine at the end of its useful life is called its scrap value. By one method of depreciation (where it is assumed a constant percentage of the value depreciates annually), the scrap value, S, is given by

$$S = C(1 - r)^n,$$

where C is the original cost, n is the useful life in years, and r is the constant percent of depreciation.

(a) Find the scrap value of a machine costing \$30,000, having a useful life of 12 years and a constant annual rate of depreciation of 15%.

(b) A machine has its value cut in half in 6 years. Find the constant annual rate of depreciation.

Solve each equation.

33. $7^{2y-1} = 1$ **34.** $8^m = 3^{m+1}$ **35.** $9^{2k} = 5^{3k-1}$ **36.** $7^{x^2+2x} = \frac{1}{7}$

37. $2^{2x^2+1} = 8^x$ **38.** $25^{2-x} = 5^{2x^2}$ **39.** $(1 + .03)^n = 90$ **40.** $100(1 + .02)^{3+n} = 150$

41. $\log_3 x + \log_3 (2x + 5) = 1$

42. $\log_2 x + \log_2 (x - 7) = 3$

43. $\log x + \log (3x - 7) = 1$

44. $\log x = 1 - \log (x - 3)$

Review Exercises

Evaluate each expression. See Section 11.2.

45. $\dfrac{\log_2 4}{\log_2 8}$ **46.** $\dfrac{\log_3 27}{\log_3 1/3}$ **47.** $\dfrac{\log .1}{\log 100}$ **48.** $\dfrac{\log 10{,}000}{\log .001}$