

Dividing Points [J10 3.1.1] pps. 125-126

■ Internal and external dividing points

When P lies between A and B, we say "P divides AB internally" and "P is an internal dividing point of AB".

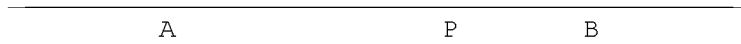


Figure 1

When P lies beyond A or B, we say "P divides AB externally" and "P is an external dividing point of AB".

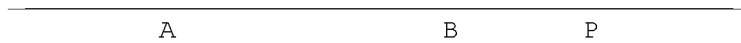


Figure 2

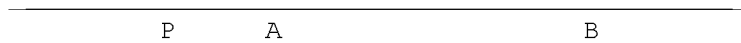


Figure 3

■ The ratio in which P divides AB

Whether the division of AB by P is internal or external, we can talk about the ratio into which P divides AB. In each case, this is the ratio of the distances of P from the endpoints of AB. In the following discussion, points A, B, P have, respectively, coordinates a, b, p.

In each of the figures above,

the ratio into which P divides AB is $\frac{\text{distance AP}}{\text{distance BP}} = \frac{|AP|}{|BP|}$.

■ Internally divided

In figure 1: $a < p < b$. So, $\frac{|AP|}{|BP|} = \frac{p-a}{b-p}$.

■ Externally divided

In figure 2: $a < b < p$. So, $\frac{|AP|}{|BP|} = \frac{p-a}{p-b}$.

In figure 3: $p < a < b$. So, $\frac{|AP|}{|BP|} = \frac{a-p}{b-p} = \frac{p-a}{p-b}$.

■ Theorems

◆ Thm 1. Given two points $A(a)$ and $B(b)$. If $P(p)$ divides AB internally in the ratio $\frac{m}{n}$, then $p = \frac{mb+na}{m+n}$.

Proof.

$$\frac{m}{n} = \frac{|AP|}{|BP|} = \frac{p-a}{b-p} \implies m(b-p) = n(p-a) \implies mb - mp = np - na \implies mb + na = np + mp \implies p = \frac{mb+na}{m+n}.$$

□

◆ Thm 2. Given two points $A(a)$ and $B(b)$. If $P(p)$ divides AB externally in the ratio $\frac{m}{n}$, then $p = \frac{mb-na}{m-n}$.

$$\frac{m}{n} = \frac{|AP|}{|BP|} = \frac{p-a}{p-b} \implies m(p-b) = n(p-a) \implies mp - mb = np - na \implies mp - np = mb - na \implies p = \frac{mb-na}{m-n}.$$

□

Note that if P is the midpoint of AB then P divides AB internally into two equal segments. Thus, $\frac{m}{n} = 1 \implies m = n$. By theorem 1,

$$p = \frac{mb+na}{m+n} = \frac{b+a}{2}$$

■ Examples

[EX1] Given $A(2)$ and $B(9)$. Find point $P(p)$ that internally divides AB at a ratio of $\frac{3}{4}$.

$$p = \frac{mb+na}{m+n}, \text{ where } \frac{m}{n} = \frac{3}{4}. \text{ Thus, } p = \frac{3(9)+4(2)}{7} = \frac{35}{7} = 5.$$

[EX2] Given $A(2)$ and $B(5)$. Find point $P(p)$ that externally divides AB at a ratio of $\frac{7}{4}$.

$$p = \frac{mb-na}{m-n}, \text{ where } \frac{m}{n} = \frac{7}{4}. \text{ Thus, } p = \frac{7(5)-4(2)}{7-4} = \frac{27}{3} = 9.$$

[EX3] Given $A(5)$ and $B(9)$. Find point $P(p)$ that externally divides AB at a ratio of $\frac{3}{7}$.

$$p = \frac{mb-na}{m-n}, \text{ where } \frac{m}{n} = \frac{3}{7}. \text{ Thus, } p = \frac{3(9)-7(5)}{3-7} = \frac{-8}{-4} = 2.$$

Dividing Points Practice Problems (with answers)

■ A. Given points $A(-3)$, $B(9)$, find the following

- [1] The number p such that point $P(p)$ that divides AB internally in the ratio 3 : 4. ANS: $\frac{15}{7}$
- [2] The number p such that point $P(p)$ that divides AB externally in the ratio 3 : 4. ANS: -39
- [3] The number p such that point $P(p)$ that divides AB internally in the ratio 2 : 5. ANS: $\frac{3}{7}$
- [4] The number p such that point $P(p)$ that divides AB externally in the ratio 2 : 5. ANS: -11

■ B. Given point $A(3)$ find the following

- [1] The number b such that point $P(2)$ divides AB internally in the ratio 2 : 5. ANS: $\frac{-1}{2}$
- [2] The number b such that point $P(5)$ divides AB externally in the ratio 2 : 5. ANS: 0
- [3] The number b such that point $P(2)$ divides AB internally in the ratio 3 : 2. ANS: $\frac{4}{3}$
- [4] The number b such that point $P(-3)$ divides AB externally in the ratio 7 : 2. ANS: $\frac{-9}{7}$