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From Calculus, James Stewart, 3rd ed., Brooks/Cole Pub.

87–94 ■ Find the functions $f \circ g$, $g \circ f$, $f \circ f$, and $g \circ g$ and their domains.

87. $f(x) = 2x^2 - x, \quad g(x) = 3x + 2$

88. $f(x) = \sqrt{x - 1}, \quad g(x) = x^2$

89. $f(x) = 1/x, \quad g(x) = x^3 + 2x$

90. $f(x) = \dfrac{1}{x - 1}, \quad g(x) = \dfrac{x - 1}{x + 1}$

91. $f(x) = \sqrt[3]{x}, \quad g(x) = 1 - \sqrt{x}$

92. $f(x) = \sqrt{x^2 - 1}, \quad g(x) = \sqrt{1 - x}$

93. $f(x) = \dfrac{x + 2}{2x + 1}, \quad g(x) = \dfrac{x}{x - 2}$

94. $f(x) = \dfrac{1}{\sqrt{x}}, \quad g(x) = x^2 - 4x$

95–98 ■ Find $f \circ g \circ h$.

95. $f(x) = x - 1, \quad g(x) = \sqrt{x}, \quad h(x) = x - 1$

96. $f(x) = \dfrac{1}{x}, \quad g(x) = x^3, \quad h(x) = x^2 + 2$

97. $f(x) = x^4 + 1, \quad g(x) = x - 5, \quad h(x) = \sqrt{x}$

98. $f(x) = \sqrt{x}, \quad g(x) = \dfrac{x}{x - 1}, \quad h(x) = \sqrt[3]{x}$

99–102 ■ Express the function in the form $f \circ g$.

99. $F(x) = (x - 9)^5$

100. $F(x) = \sqrt{x} + 1$

101. $G(x) = \dfrac{x^2}{x^2 + 4}$

102. $G(x) = \dfrac{1}{x + 3}$

103–104 ■ Express the function in the form $f \circ g \circ h$.

103. $H(x) = \dfrac{1}{x^2 + 1}$

104. $H(x) = \sqrt[3]{\sqrt{x} - 1}$

105. A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 60 cm/s. Express the area of this circle as a function of time t (in seconds).

106. A spherical balloon is being inflated. If the radius of the balloon is increasing at a rate of 1 cm/s, express the volume of the balloon as a function of time t (in seconds).

107. If $f(x) = 3x + 5$ and $h(x) = 3x^2 + 3x + 2$, find a function g such that $f \circ g = h$.

108. If $f(x) = x + 4$ and $h(x) = 4x - 1$, find a function g such that $g \circ f = h$.

109. Let $f(x) = 1/x$ and $g(x) = x$. How does $f \circ f$ differ from g?

2 TYPES OF FUNCTIONS; SHIFTING AND SCALING

In solving calculus problems you will find that it is helpful to be familiar with the graphs of some commonly occurring functions. We classify various types of functions as follows.

CONSTANT FUNCTIONS The constant function $f(x) = c$ has domain $\mathbb{R}$ and its range consists of the single number c. Its graph is a horizontal line and is illustrated in Figure 1 for $c = 2$.

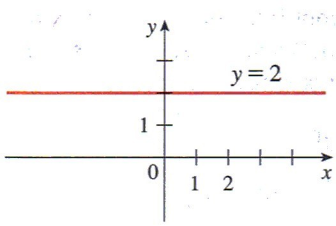


FIGURE 1

POWER FUNCTIONS A function of the form $f(x) = x^a$, where a is a constant, is called a **power function**. We consider several cases.

(a) $a = n$, a positive integer.

The graphs of $f(x) = x^n$ for $n = 1, 2, 3, 4$, and 5 are shown in Figure 2. We already know the shape of the graphs of $y = x$ (a line through the origin with slope 1), $y = x^2$ (a parabola), and $y = x^3$ (Example 7 in Section 1).

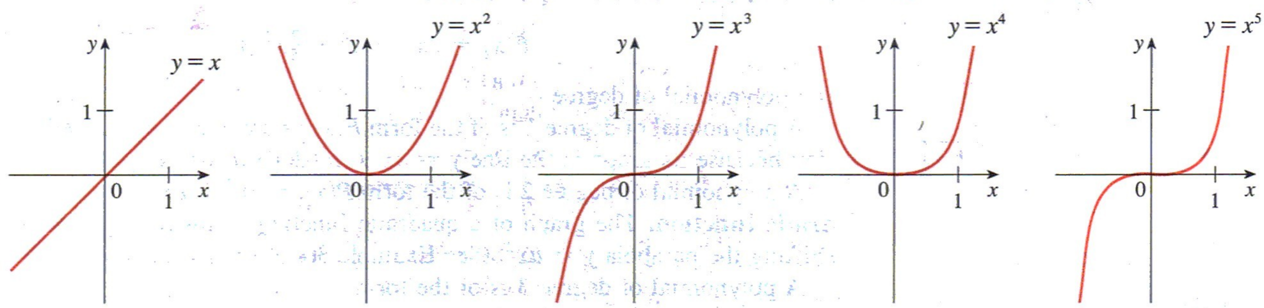


FIGURE 2 Graphs of $f(x) = x^n$, $n = 1, 2, 3, 4, 5$

The general shape of the graph of $f(x) = x^n$ depends on whether n is even or odd. If n is even, then $f(x) = x^n$ is an even function and its graph is similar to the parabola $y = x^2$. If n is odd, then $f(x) = x^n$ is an odd function and its graph is similar to that of $y = x^3$. Notice from Figure 3, however, that as n increases, the graph of $y = x^n$ becomes flatter near 0 and steeper when $|x| \geqslant 1$. (If x is small, then x^2 is smaller, x^3 is even smaller, x^4 is smaller still, and so on.)

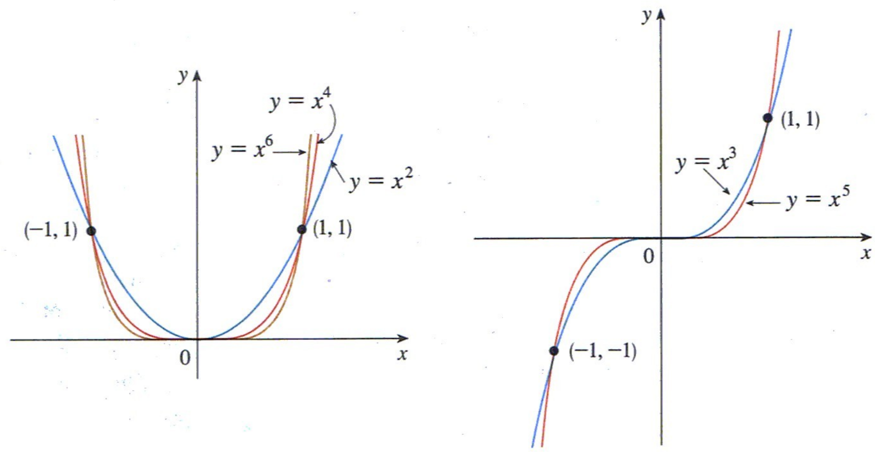


FIGURE 3

(b) $a = -1$.
The graph of the reciprocal function $f(x) = x^{-1} = 1/x$ is shown in Figure 4. Its graph has the equation $y = 1/x$ or $xy = 1$. This is an equilateral hyperbola with the coordinate axes as its asymptotes.

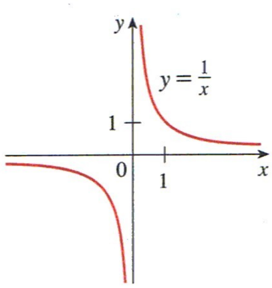
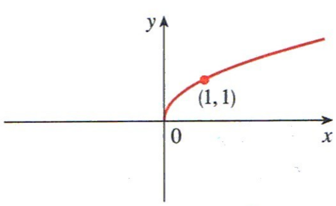
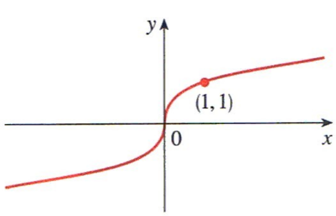


FIGURE 4

(c) $a = 1/n$, n a positive integer.
The function $f(x) = x^{1/n} = \sqrt[n]{x}$ is a **root function**. For $n = 2$ it is the square root function $f(x) = \sqrt{x}$ whose domain is $[0, \infty)$ and whose graph is the upper half of the parabola $x = y^2$ [see Figure 5(a)]. For other even values of n, the graph of $y = \sqrt[n]{x}$ is similar to that of $y = \sqrt{x}$. For $n = 3$ we have the cube root function $f(x) = \sqrt[3]{x}$ whose domain is $\mathbb{R}$ (recall that every real number has a cube root) and whose graph is shown in Figure 5(b). The graph of $y = \sqrt[n]{x}$ for n odd $(n > 3)$ is similar to that of $y = \sqrt[3]{x}$.



(a) $f(x) = \sqrt{x}$



(b) $f(x) = \sqrt[3]{x}$

FIGURE 5
Graphs of root functions

POLYNOMIALS A function P is called a **polynomial** if

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where n is a nonnegative integer and the numbers $a_0, a_1, a_2, \ldots, a_n$ are constants called the **coefficients** of the polynomial. The domain of any polynomial is $\mathbb{R} = (-\infty, \infty)$. If the leading coefficient $a_n \neq 0$, then the **degree** of the polynomial is n. For example, the function

$$P(x) = 2x^6 - x^4 + \tfrac{2}{5}x^3 + \sqrt{2}$$

is a polynomial of degree 6.

A polynomial of degree 1 is of the form $P(x) = ax + b$ and is called a **linear function** because its graph is the line $y = ax + b$ (slope a, y-intercept b).

A polynomial of degree 2 is of the form $P(x) = ax^2 + bx + c$ and is called a **quadratic function**. The graph of a quadratic function is always a parabola obtained by shifting the parabola $y = ax^2$. (See Example 3.)

A polynomial of degree 3 is of the form

$$P(x) = ax^3 + bx^2 + cx + d$$

and is called a **cubic function**. Figure 6 shows the graph of a cubic function in part (a) and graphs of polynomials of degrees 4 and 5 in parts (b) and (c). We will see later why the graphs have these shapes.

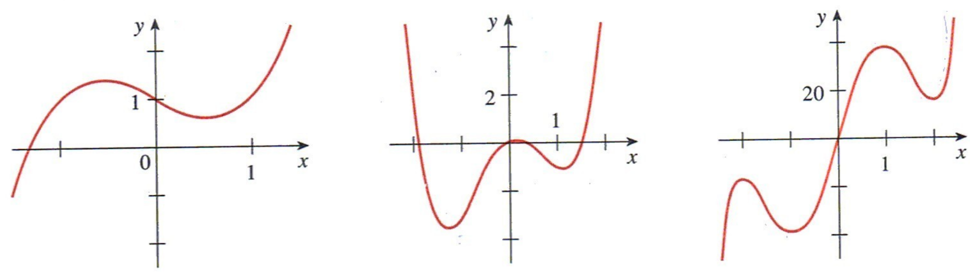


FIGURE 6 (a) $y = x^3 - x + 1$ (b) $y = x^4 - 3x^2 + x$ (c) $y = 3x^5 - 25x^3 + 60x$

Polynomials are commonly used to model various quantities that occur in the natural and social sciences. For instance, in Section 2.3 we will explain why economists often use a polynomial $P(x)$ to represent the cost of producing x units of a commodity.

RATIONAL FUNCTIONS A **rational function** f is a ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. The domain consists of all values of x such that $Q(x) \neq 0$. For example, the function

$$f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$$

is a rational function with domain $\{x \mid x \neq \pm 2\}$. Its graph is shown in Figure 7.

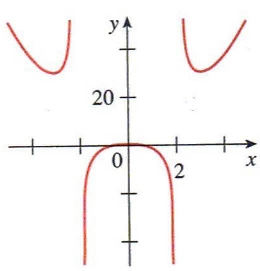


FIGURE 7
$f(x) = \dfrac{2x^4 - x^2 + 1}{x^2 - 4}$

ALGEBRAIC FUNCTIONS A function f is called an **algebraic function** if it can be constructed using algebraic operations (addition, subtraction, multiplication, division, and taking roots) starting with polynomials. Any rational function is automatically an algebraic function. Here are two more examples:

$$f(x) = \sqrt{x^2 + 1} \qquad g(x) = \frac{x^4 - 16x^2}{x + \sqrt{x}} + (x - 2)\sqrt[3]{x + 1}$$

When we sketch algebraic functions in Chapter 3 we will see that their graphs can assume a variety of shapes. Figure 8 illustrates some of the possibilities.

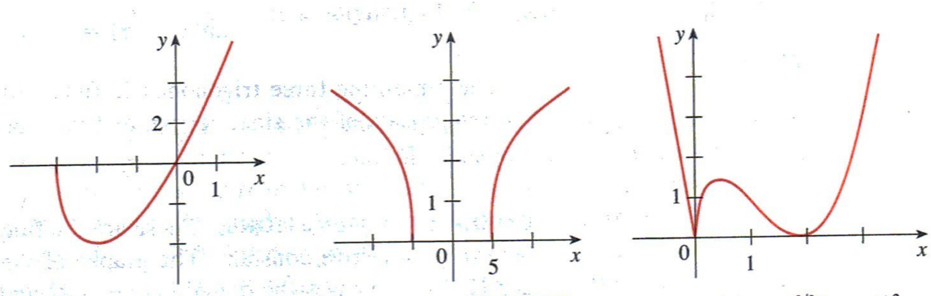
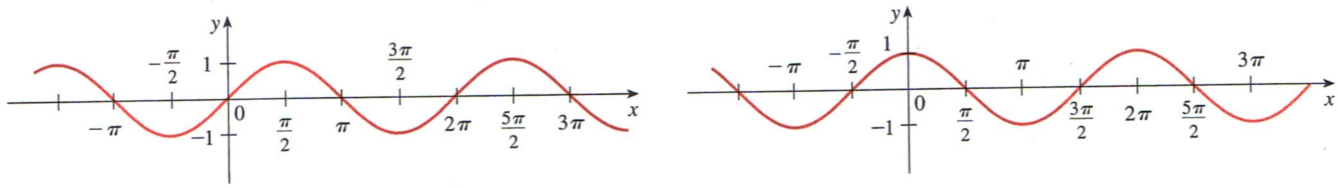


FIGURE 8 (a) $f(x) = x\sqrt{x + 3}$ (b) $g(x) = \sqrt[4]{x^2 - 25}$ (c) $h(x) = x^{2/3}(x - 2)^2$

TRIGONOMETRIC FUNCTIONS Trigonometry and the trigonometric functions are reviewed on the endpapers and in Appendix D. In calculus the convention is that radian measure is always used (except when otherwise indicated). For example, when we use the function $f(x) = \sin x$, it is understood that $\sin x$ means the sine of the angle whose radian measure is x. Thus the graphs of the sine and cosine functions are as shown in Figure 9.



(a) $f(x) = \sin x$

(b) $g(x) = \cos x$

FIGURE 9

Notice that for both the sine and cosine functions the domain is $(-\infty, \infty)$ and the range is the closed interval $[-1, 1]$. Thus, for all values of x, we have

$$-1 \leqslant \sin x \leqslant 1 \qquad -1 \leqslant \cos x \leqslant 1$$

Also, the zeros of the sine function occur at the integer multiples of π; that is,

$$\sin x = 0 \quad \text{when} \quad x = n\pi \quad n \text{ an integer}$$

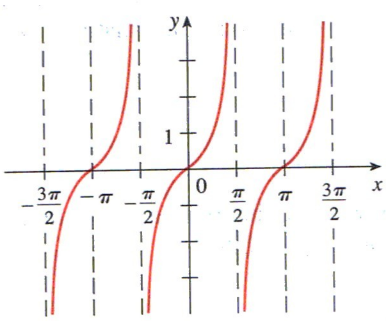
An important property of the sine and cosine functions is that they are periodic functions and have period 2π. This means that, for all values of x,

$$\sin(x + 2\pi) = \sin x \qquad \cos(x + 2\pi) = \cos x$$

The periodic nature of these functions makes them suitable for modeling periodic phenomena such as tides, vibrating springs, and sound waves.

The tangent function is related to the sine and cosine functions by the equation

$$\tan x = \frac{\sin x}{\cos x}$$



$y = \tan x$

FIGURE 10

and its graph is shown in Figure 10. It is undefined when $\cos x = 0$, that is, when $x = \pm\pi/2, \pm 3\pi/2, \ldots$. Its range is $(-\infty, \infty)$. Notice that the tangent function has period π:

$$\tan(x + \pi) = \tan x \qquad \text{for all } x$$

The remaining three trigonometric functions (cosecant, secant, and cotangent) are the reciprocals of the sine, cosine, and tangent functions. Their graphs are shown in Appendix D.

EXPONENTIAL FUNCTIONS These are the functions of the form $f(x) = a^x$, where the base a is a positive constant. The graphs of $y = 2^x$ and $y = (0.5)^x$ are shown in Figure 11. In both cases the domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.

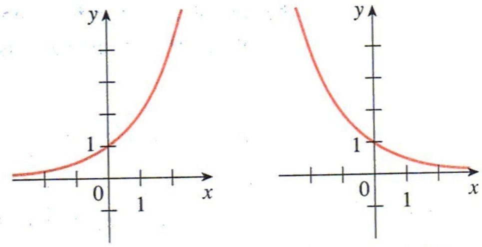


FIGURE 11 (a) $y = 2^x$ (b) $y = (0.5)^x$

Exponential functions will be studied in detail in Chapter 3 and we will see that they are useful for modeling population growth if $a > 1$ and radioactive decay if $a < 1$.

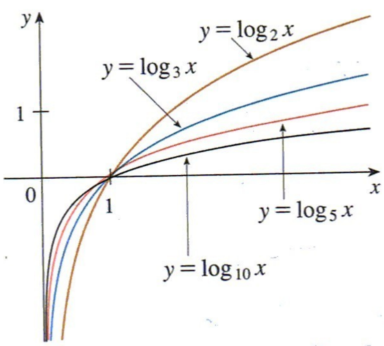


FIGURE 12

LOGARITHMIC FUNCTIONS These are the functions $f(x) = \log_a x$, where the base a is a positive constant. They are the inverse functions of the exponential functions and will also be studied in Chapter 3. Figure 12 shows the graphs of four logarithmic functions with various bases. In each case the domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the function increases slowly when $x > 1$.

TRANSCENDENTAL FUNCTIONS These are functions that are not algebraic. The set of transcendental functions includes the trigonometric, inverse trigonometric, exponential, and logarithmic functions, but it also includes a vast number of other functions that have never been named. In Chapter 10 we will study transcendental functions that are defined as sums of infinite series.

EXAMPLE 1 Classify the following functions as one of the types of functions that we have discussed.

(a) $f(x) = 5^x$

(b) $g(x) = x^5$

(c) $h(x) = \dfrac{1 + x}{1 - \sqrt{x}}$

(d) $u(t) = 1 - t + 5t^4$

SOLUTION

(a) $f(x) = 5^x$ is an exponential function. (The x is the exponent.)

(b) $g(x) = x^5$ is a power function. (The x is the base.)

(c) $h(x) = \dfrac{1 + x}{1 - \sqrt{x}}$ is an algebraic function.

(d) $u(t) = 1 - t + 5t^4$ is a polynomial of degree 4. ■

TRANSFORMATIONS OF FUNCTIONS

By applying certain transformations to the graph of a given function we can obtain the graphs of certain related functions and thereby reduce the amount of work in graphing. Let us first consider **translations**. By adding the constant function $g(x) = c > 0$ to a given function f by graphical addition, we see that the graph of $y = f(x) + c$ is just the graph of $y = f(x)$ shifted upward a distance of c units. Likewise, if $g(x) = f(x - c)$, where $c > 0$, then the value of g at x is the same as the value of f at $x - c$ (c units to