[06-12-19-T11]

Extension of exponents

■ Positive integer exponent

When n is a positive integer, a^n is easy to understand. We say

$$a^n = \underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{\text{n factors of } a}$$

When n is a negative integer, zero, a rational number, or an irrational number it is not so obvious what the symbol a^n means. Here, you will learn the definitions for such cases. Ideally, you will see why the definitions given are reasonable and useful.

■ a^0

Consider the rule $a^m \cdot a^n = a^{m+n}$ when $n = 0$.

$$a^m \cdot a^0 = a^{m+0} = a^m$$

Notice that a^0 behaves just as does 1, the multiplicative identity element.

◆ Definition a^0

For any real number a other than zero,

$$a^0 = 1$$

■ a^n, $n \in \{-1, -2, -3, \ldots\}$.

Let n be a positive integer, in which case $-n$ is a negative integer. Then,

$$a^n \cdot a^{-n} = a^{n+(-n)} = a^0 = 1.$$

a^{-n} behaves just as does $\frac{1}{a^n}$ the multiplicative inverse element of a^n.

◆ Definition a^n, $n \in \mathbb{Z}^-$

For any real number $a \neq 0$ and positive integer n (NB: if n is positive, $-n$ is negative),

$$a^{-n} = \frac{1}{a^n}$$

■ **$a^{\frac{1}{n}}$, $n \in \{1, 2, 3, \ldots\}$**

Suppose, for example, that $n = 2$. Then,

$$a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^{1} = a.$$

$a^{\frac{1}{2}}$ behaves just as does $\sqrt{a}$ the square root of a.

◆ Definition $a^{\frac{1}{n}}$

a. If n is even positive integer and $a > 0$, then $a^{\frac{1}{n}} = \sqrt[n]{a}$

b. If n is odd positive integer, then $a^{\frac{1}{n}} = \sqrt[n]{a}$

c. If n is a positive integer, then $0^{\frac{1}{n}} = 0$

■ **$a^{\frac{m}{n}}$, $m, n \in \{1, 2, 3, \ldots\}$**

Since $a^{\frac{1}{n}} = \sqrt[n]{a}$ and $a^{\frac{m}{n}} = a^{\frac{1}{n} \cdot m}$, consider $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^{m}$

$$\left(\sqrt[n]{a}\right)^{m} = \underbrace{\sqrt[n]{a} \cdot \sqrt[n]{a} \cdot \sqrt[n]{a} \cdot \ldots \cdot \sqrt[n]{a}}_{m \text{ factors of } \sqrt[n]{a}}$$

◆ Definition $a^{\frac{m}{n}}$

If m, n are positive integers, then $a^{\frac{m}{n}} = \sqrt[n]{a^{m}} = \left(\sqrt[n]{a}\right)^{m}$, providing $\sqrt[n]{a}$ exists.

■ **a^{x}, $x \in \mathbb{R}$**

Defining a^{x}, $x \in \mathbb{R}$ depends on giving a definition of a^{x} when x is irrational. A completely satisfactory definition requires an understanding of real numbers that you have not yet acquired. We can say this much: you know how to approximate an irrational number by a rational number to any desired accuracy (e.g. newton's method). Perhaps you will accept, for the time being, the idea that when b is a positive real number and x is any real number, the symbol b^{x} names a unique real number. And, while you are in the mood to buy things, let's throw in that for any positive real number b, $b \neq 1$, and any positive real number k, there is a number x such that $k = b^{x}$.

■ **Notes**

[1] 0^{0} is undefined.

[2] We have been careful to give definitions that are consistent with the familiar rules for exponents; hence, those rules hold for integer, rational, and real exponents.