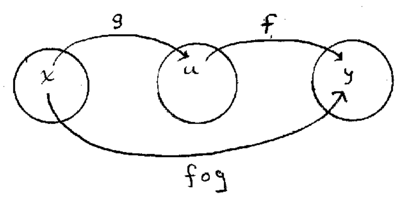
■ Composition

Suppose

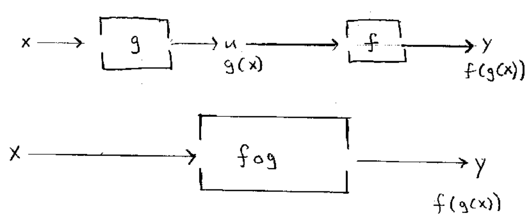
$u = g(x) = x^2 + 1$, and

$y = f(u) = \sqrt{u}$.

Then y is a function of u and u is, in turn, a function of x. So, ultimately, y is a function of x. We compose a new function, $f \circ g$, from the two functions f and g. We call this new function the composition of f and g, and we write it like this: $\boldsymbol{f \circ g}$.



You can also picture the composition of f and g like this



In this example,

$$y = (f \circ g)(x) = f(g(x)) = f(x^2 + 1) = \sqrt{x^2 + 1} .$$

◆ **Def. Given two functions f and g, the composite function $f \circ g$ is defined by**

$$(f \circ g)(x) = f(g(x))$$

Where the domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

■ Example 1

For $f : f(x) = x^2 + 1$ and $g : g(x) = 3x - 7$, find $f \circ g$ and $g \circ f$.

Solution

$f \circ g$ is defined by $f(g(x))$. $f(g(x)) = f(3x - 7) = (3x - 7)^2 + 1 = 9x^2 - 42x + 49 + 1 = 9x^2 - 42x + 50$. $\mathcal{D} = \mathbb{R}$.

$g \circ f$ is defined by $g(f(x))$. $g(f(x)) = g(x^2 + 1) = 3(x^2 + 1) - 7 = 3x^2 + 3 - 7 = 3x^2 - 4$. $\mathcal{D} = \mathbb{R}$.

Notice that $f \circ g \neq g \circ f$. Composition of functions is *not* a commutative operation.

The order of the composition matters.

■ Example 2

If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2 - x}$, find $f \circ f$ and $g \circ f$.

Solution

$(f \circ f)(x) = f(f(x)) = f\left(\sqrt{x}\right) = \sqrt{\sqrt{x}} = \sqrt[4]{x}$. The domain of $f \circ f$ is $[0, \infty)$.

$(g \circ f)(x) = g(f(x)) = \sqrt{2 - f(x)} = \sqrt{2 - \sqrt{x}}$. What's the domain of $g \circ f$? $\sqrt{x}$ is a real number when $x \geq 0$. $\sqrt{2 - \sqrt{x}}$ is a real number when $\sqrt{x} \leq 2 \Longrightarrow x \leq 4$. Thus, the domain of $g \circ f$ is $[0, 4]$.

Many times, you will need to decompose a function.

■ Example 3

If $h(x) = (x + 5)^2$, find functions f and g such that $h = f \circ g$.

Solution

$g(x) = x + 5$ and $f(x) = x^2$.

■ Example 4

If $h(x) = 2x^2 - 3$ and $f(x) = 2x + 1$, find the function g such that $h = f \circ g$.

Solution

$h = f \circ g \Longrightarrow h(x) = f(g(x)) = 2(g(x)) + 1$

$h(x) = 2x^2 - 3$

so, $2(g(x)) + 1 = 2x^2 - 3$.

Solving for $g(x)$, you get

$g(x) = x^2 - 2$.

You should check this answer:

$f(g(x)) = 2(x^2 - 2) + 1 = 2x^2 - 4 + 1 = 2x^2 - 3$. OK.