[06-08-28-T10]

Quadratic Inequalities

Form of quadratic inequality

A quadratic inequality has the form $\boldsymbol{a x^2 + b x + c \gtreqless 0,\ a > 0}$. Note that $a > 0$. If in a given case $a < 0$, then multiply both sides of $a x^2 + b x + c \gtreqless 0$ by -1, remembering that the inequality is reversed by so doing.

Solutions

It is easiest to talk about the solutions of quadratic inequalities if we consider three cases. Note $D = b^2 - 4 a c$.

Case 1: $D > 0$.

Case 2: $D = 0$.

Case 3: $D < 0$.

■ Case 1. $D > 0$

Since $D > 0$, there are two real roots, α and β, of $a x^2 + b x + c = 0$. Thus, the factors of $f(x) = a x^2 + b x + c$ are $(x - \alpha)$ and $(x - \beta)$. Suppose, WLOG that $\alpha < \beta$. Then,

x	$x < \alpha$	α	$\alpha < x < \beta$	β	$x > \beta$
$x - \alpha$	$-$	0	$+$	$+$	$+$
$x - \beta$	$-$	$-$	$-$	0	$+$
$f(x) = (x - \alpha)(x - \beta)$	$+$	0	$-$	0	$+$

[1] $a x^2 + b x + c > 0$. *Solution*: $x < \alpha \vee x > \beta$. Equivalent $(-\infty, \alpha) \bigcup (\beta, \infty +)$. Equivalent $\{x \ni x < \alpha\} \bigcup \{x \ni x > \beta\}$.

[2] $a x^2 + b x + c \geq 0$. *Solution*: $x \leq \alpha \vee x \geq \beta$. Equivalent $(-\infty, \alpha] \bigcup [\beta, \infty +)$. Equivalent $\{x \ni x \leq \alpha\} \bigcup \{x \ni x \geq \beta\}$.

[3] $a x^2 + b x + c < 0$. *Solution*: $\alpha < x < \beta$. Equivalent (α, β). Equivalent $\{x \ni \alpha < x < \beta\}$.

[4] $a x^2 + b x + c \leq 0$. *Solution*: $\alpha \leq x \leq \beta$. Equivalent $[\alpha, \beta]$. Equivalent $\{x \ni \alpha \leq x \leq \beta\}$.

■ Case 2. $D = 0$

x	$x < \alpha$	α	$x > \alpha$
$x - \alpha$	$-$	0	$+$
$f(x) = (x - \alpha)^2$	$+$	0	$+$

[1] $a x^2 + b x + c > 0$. *Solution*: $\mathbb{R} - \{\alpha\}$. Equivalent $(-\infty, \alpha) \bigcup (a, \infty +)$. Equivalent $\{x \ni x \in \mathbb{R} \wedge x \neq \alpha\}$.

[2] $a x^2 + b x + c \geq 0$. *Solution*: $\mathbb{R}$. Equivalent $(-\infty, \infty +)$. Equivalent $\{x \ni x \in \mathbb{R}\}$.

[3] $a x^2 + b x + c < 0$. *Solution*: ϕ.

[4] $a x^2 + b x + c \leq 0$. *Solution*: α. Equivalent $\{x \ni x = \alpha\}$.

■ Case 3. $D < 0$

Since $D < 0$, there is no real number x such that $a x^2 + b x + c = 0$.

$$
\begin{aligned}
&a x^2 + b x + c \\
&= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{a b^2}{4a^2} \\
&= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} \\
&= a\left(x + \frac{b}{2a}\right)^2 + \frac{-(b^2 - 4ac)}{4a} \\
&= a\left(x + \frac{b}{2a}\right)^2 + \frac{-D}{4a}
\end{aligned}
$$

Now, $a > 0$ and certainly $\left(x + \frac{b}{2a}\right)^2 > 0$. Since $D < 0$, $-D > 0$. Thus, $a\left(x + \frac{b}{2a}\right)^2 + \frac{-D}{4a} > 0$. Therefore, $a x^2 + b x + c > 0$ for all real values of *x*. To summarize,

[1] $a x^2 + b x + c > 0$. *Solution*: $\mathbb{R}$. Equivalent $(-\infty, \infty +)$. Equivalent $\{x \ni x \in \mathbb{R}\}$.

[2] $a x^2 + b x + c \geq 0$. *Solution*: $\mathbb{R}$. Equivalent $(-\infty, \infty +)$. Equivalent $\{x \ni x \in \mathbb{R}\}$.

[3] $a x^2 + b x + c < 0$. *Solution*: ϕ.

[4] $a x^2 + b x + c \leq 0$. *Solution*: ϕ.

Advice

It would be well to remember the results summarized above for all three cases of **D**. Suppose you wish to solve a particular quadratic inequality, but you have forgotten this summary. It might be easiest to derive the result in the relevant case $D < 0$, $D = 0$, $D > 0$ in general, then apply the specific roots for the particular inequality with which you are working. Deriving the general result for cases $D > 0$ and $D = 0$ is straightforward. Case $D < 0$ seems a little more difficult. It is, however, be the easiest case of the three. Remember that when $D < 0$, the graph of $f(x) = a x^2 + b x + c$ never touches the *x*-axis. Indeed, it lies entirely above the *x*-axis.