[07-01-29-P2-8]

Exercises [A-1]

In exercises 1–6 find the slope, x-intercept, y-intercept for each line. Use the intercepts to make a rough sketch of each graph. Check the slope.

1. $4x - y = 4$
2. $3x + 5y = 15$
3. $y = \frac{1}{2}x + \frac{5}{2}$
4. $2x + y + 8 = 0$
5. $3x + 2y = 6$
6. $3x - 2y - 12 = 0$
7. Find the equation of the line with slope 3 and y-intercept 4.
8. Find the equation of the line with slope $-\frac{1}{2}$ and y-intercept 3.
9. Find the equation of the line through $(0, -3)$ with slope 2.
10. Find the equation of the line through $(0, 0)$ with slope 1.5.
11. Consider the equations $y = 2.5x$ and $y = 2.5x + 5$.
    (a) In which equation does y vary directly as x?
    (b) Which equation has a graph passing through the origin?
12. If $y = mx$, where m is a positive constant, y varies directly as x. Draw graphs illustrating this relationship when (a) $m = \frac{1}{2}$, (b) $m = 1$, (c) $m = 5$.
13. What point lies on the graph of every equation representing a direct-variation relationship between y and x?
14. The graph of a direct-variation relationship between y and x has slope 4.2. Find the value of y when $x = 15$.

Find the equations of the straight lines determined by the given conditions in exercises 15–25.

15. The line through $(1, 3)$ with slope 1.
16. The line through $(5, -10)$ with slope -2.
17. The line through $(-2, 0)$ with slope $\frac{5}{2}$.
18. The line through $(0, 0)$ with slope $-\frac{2}{3}$.
19. The line through $(-6, 2)$ and $(10, 10)$.
20. The line through $(0, -4)$ and $(6, 0)$.
21. The line through $(3, 6)$ parallel to the x-axis.
22. The line through $(-2, 3)$ parallel to $2x - y = 6$.
23. The line through $(3, -2)$ perpendicular to $x + 2y - 12 = 0$.
24. The line through $(6, 4)$ perpendicular to $y = 3x - 5$.
25. The line through $(-2, -1)$ parallel to $3x - 2y + 2 = 0$.
26. A, B have coordinates $(1, 6)$, $(3, 0)$. (a) Find the coordinates of M the midpoint of AB. (b) Find the equation of the line through M perpendicular to AB.

27. (a) Find the equation of the perpendicular bisector of the line segment joining $A(3, 8)$ and $B(-1, 2)$.
    (b) If the perpendicular bisector meets the x-axis at P, find the lengths PA and PB.

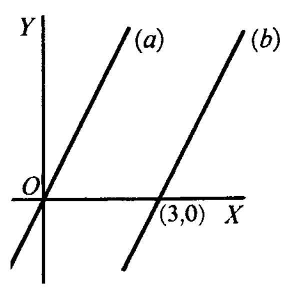
28. (a) Find the equation of the perpendicular bisector of the line segment joining $A(1, -3)$ and $B(5, 5)$.
    (b) If the perpendicular bisector meets the y-axis at P, find the lengths PA and PB.

Exercises [A-2]

In exercises 1–6 find the slope, x-intercept, y-intercept for each line. Use the intercepts to make a rough sketch of each graph. Check the slope.

1. $y = 2x - 8$
2. $x + 3y = 9$
3. $5x - 2y + 20 = 0$
4. $x + y = 6$
5. $y = \frac{2}{3}x + 2$
6. $3x + 2y - 4 = 0$

7. Find the equation of the line with slope 2 and y-intercept -3.
8. Find the equation of the line with slope $-\frac{2}{3}$ and y-intercept 2.
9. Find the equation of the line with slope $-\frac{1}{2}$ and y-intercept 0.
10. Which of the graphs (a) and (b) represents a direct-variation relationship between y and x? Find the equations of the graphs if the slope of each is 2.



11. The graph of a direct-variation relationship between y and x passes through the points (x_1, y_1), (x_2, y_2). Show that

$$\frac{y_1}{x_1} = \frac{y_2}{x_2} = \frac{y_2 - y_1}{x_2 - x_1}$$

Find the equations of the straight lines determined by the given conditions in exercises 12–24.

12. The line through $(-4, 2)$ with slope 1.
13. The line through $(-3, 2)$ with slope $-\frac{2}{3}$.
14. The line through $(0, 6)$ with slope $\frac{3}{2}$.
15. The line through $(-\frac{1}{2}, -2)$ with slope 2.
16. The line through $(-2, 1)$ and $(8, 6)$.
17. The line through $(3, 0)$ and $(0, 2)$.
18. The line through the origin with slope -2.
19. The line through $(0, -4)$ and $(6, -4)$.

20. The line through $(5, 0)$ and $(5, 8)$.
21. The line through $(-3, 2)$ parallel to $x + 3y = 0$.
22. The line through $(-2, -1)$ perpendicular to $x + 4y = 0$.
23. The line through $(4, 3)$ perpendicular to $2x + 3y + 6 = 0$.
24. The line through $(3, -1.5)$ perpendicular to $2x - 5y - 14 = 0$.
25. (a) Find the equation of the perpendicular bisector of the line segment joining $A(1, 3)$ and $B(-3, 5)$.
    (b) If the perpendicular bisector meets the y-axis at P, find the lengths PA and PB.
26. (a) Find the equation of the perpendicular bisector of the line segment joining $A(2, 3)$ and $B(4, -5)$.
    (b) If the perpendicular bisector meets the x-axis at P, find the lengths PA and PB.

Point of Intersection of Two Lines

If two lines l_1 and l_2 intersect at A, then A is a member of the set of points that form l_1 and also a point of the set forming l_2. The coordinates of A must therefore satisfy both the equation of l_1 and the equation of l_2. The coordinates of the point of intersection may be obtained by solving the equations as a system.

In solving systems of two equations we have seen that the equations may be

1. Consistent, that is, satisfied by a single pair of values of x and y;
2. Inconsistent, that is, having no common solution;
3. Dependent, that is, equivalent, so that any solution of one is also a solution of the other.

The corresponding geometric relations between l_1 and l_2 are

1. l_1 intersects l_2;
2. l_1 is parallel to l_2;
3. l_1 coincides with l_2.

Example 1. Show in two ways that the line $3x - 6y = 5$ is parallel to the line $y = \frac{1}{2}x + 2$.

Solution:

METHOD 1. Compare the slopes of the lines.

The first equation may be written $y = \frac{1}{2}x - \frac{5}{6}$.

Comparing this with $y = \frac{1}{2}x + 2$, we see that the lines have equal slopes and different y-intercepts.

The lines are therefore parallel.

METHOD 2. Solve the equations as a system.

In the first equation replace y by $\frac{1}{2}x + 2$, and we have

$$3x - 6(\tfrac{1}{2}x + 2) = 5$$
$$3x - 3x - 12 = 5$$
$$-12 = 5.$$

The result shows that the equations are inconsistent, that is, they have no common solution. The lines therefore have no common point and are parallel.

Example 2. Find the equation of the line which is concurrent with the lines $3x - 2y - 8 = 0$ and $2x + y - 3 = 0$, and is perpendicular to the line $4x + 3y = 0$.

Solution: Three lines are concurrent if they have a point in common. Hence the required line passes through the point of intersection of $3x - 2y - 8 = 0$ and $2x + y - 3 = 0$.

By solving the equations as a system, the point of intersection is found to be $(2, -1)$.

The equation $4x + 3y = 0$ may be written $y = -\frac{4}{3}x$, so the slope of the line is $-\frac{4}{3}$.

$\therefore$ The slope of a line perpendicular to $y = -\frac{4}{3}x$ is $\frac{3}{4}$.

The required line must pass through $(2, -1)$ and have slope $\frac{3}{4}$. Hence its equation is

$$\frac{y+1}{x-2} = \frac{3}{4} \quad \text{or} \quad 3x - 4y - 10 = 0.$$

Exercises [A–1]

In exercises 1–8, state whether the lines intersect or are parallel. If they intersect, find the point of intersection, and state whether or not the lines are perpendicular.

1. $x - 2y - 1 = 0$; $2x + y - 8 = 0$.
2. $3x + 2y - 8 = 0$; $6x + 4y - 5 = 0$.
3. $x + 3y - 2 = 0$; $x - 3y - 8 = 0$.
4. $x + 3 = 0$; $x - 1 = 0$.
5. $2x - 3y - 4 = 0$; $4x + 6y + 8 = 0$.
6. $y - 4 = 0$; $2x + 5y - 12 = 0$.
7. $x + 2 = 0$; $x + 2y = 0$.
8. $x - 3 = 0$; $y + 2 = 0$.
9. Find the equation of the line which passes through $(4, -3)$ and the intersection of the lines $3x - y - 1 = 0$ and $x + 2y - 12 = 0$.
10. Find the equation of the line which passes through $(-1, 4)$ and the intersection of the lines $x + y - 12 = 0$ and $x - y - 2 = 0$.

11. Show that the three lines $x + 3y - 19 = 0$, $x - 2y - 4 = 0$, and $2x - 5y - 5 = 0$ are concurrent. (Find the point of intersection of two of the lines and show that this point is on the third line.)

12. (a) Find the equation of the line which passes through $(-1, 2)$ and is perpendicular to $3x - 4y - 4 = 0$.
    (b) Find the point of intersection of the two lines and hence find the distance of the point $(-1, 2)$ from the line $3x - 4y - 4 = 0$.

13. $\triangle ABC$ has vertices $A(-2, 0)$, $B(6, 0)$, $C(0, 8)$. Find the equations of the altitudes from A to BC and from B to AC. Show that their point of intersection is on the third altitude.

14. Find the equations of the medians of $\triangle ABC$ in exercise 13. Show that the medians are concurrent at $(\frac{4}{3}, \frac{8}{3})$.

15. (a) Find the vertices of the triangle whose sides are segments of the lines $x + 4y - 4 = 0$, $4x - 3y + 3 = 0$, $5x + y - 20 = 0$.
    (b) Find the equation of that altitude which has a negative slope.

Exercises [A-2]

1. Find the point of intersection of the lines $4x - 3y - 8 = 0$, $2x + 6y + 1 = 0$.

2. Find the point of intersection of the lines $5x + y - 10 = 0$, $4x - 2y + 15 = 0$.

3. At what point does the line joining (4, 3) to the origin meet the line $2x + 2y - 7 = 0$?

4. Find the equation of the line which passes through $(-5, 2)$ and the intersection of the lines $x + 3y = 0$ and $4x - y - 13 = 0$.

5. Show that the lines $2x - 3y + 5 = 0$, $x - y + 7 = 0$, $3x - 4y + 12 = 0$ are concurrent.

6. (a) Find the equation of the perpendicular from $(5, -1)$ to the line $2x + 3y - 20 = 0$.
   (b) Find the point of intersection of the two lines and hence obtain the distance from $(5, -1)$ to $2x + 3y - 20 = 0$.

7. Show that for all values of a and b except $a = 1$ or -1, the point of intersection of the lines $ax + y = b$, $x + ay = b$ is on the line $y = x$.

8. Using the equations $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, obtain a formula for the point of intersection of two straight lines.

9. Show that if the lines in exercise 8 have no point of intersection then $a_1b_2 = a_2b_1$.

Exercises 10–15 refer to $\triangle ABC$ with vertices $A(-2, 0)$, $B(0, 8)$, $C(4, 2)$.

10. (a) Find the equations of the three medians of the triangle ABC.
(b) Show that the three medians are concurrent at some point G.

11. If M is the midpoint of AB, and G the meeting point of the medians in exercise 10, show that $CG = 2\,GM$.

12. (a) Find the equations of the three altitudes of the triangle ABC.
(b) Show that the three altitudes are concurrent at some point H.

13. (a) Find the equations of the three perpendicular bisectors of the sides of the triangle ABC.
(b) Show that the three perpendicular bisectors are concurrent at some point P.

14. Show that P in exercise 13 is equidistant from A, B, C.

15. Show that G, H, and P (exercises 10, 12, 13) are collinear.

Exercises [B]

1. (a) Write the distance of any point $P(x, y)$ from $A(4, 0)$ and from $B(0, 2)$.
(b) Find the equation of the locus of P under the condition $PA = PB$.
(c) Verify that this locus is a line perpendicular to AB through the midpoint of AB.

2. Find the equation of the locus of points (a) equidistant from $(-1, 2)$ and $(3, 8)$, (b) equidistant from $(a, 0)$ and $(0, b)$.

3. Show that the perpendicular bisectors of the sides of any triangle are concurrent. (Let the vertices be $(0, 0)$, $(a, 0)$, (b, c).)

4. Show that the medians of the triangle with vertices $(0, 0)$, $(a, 0)$, (b, c) are concurrent.

5. Show that for any value of k the equation

$$(x + 2y + 3) + k(3x - 2y - 1) = 0$$

is that of a line through the intersection of the lines $x + 2y + 3 = 0$ and $3x - 2y - 1 = 0$. Find the value of k so that the line shall be perpendicular to $y = \frac{2}{5}x$.

6. Use the method indicated in exercise 5 to obtain the equations of the altitudes of the triangle determined by the lines $x + 4y - 4 = 0$, $4x - 3y + 3 = 0$, $5x + y - 20 = 0$.

7. (a) Write the slope (m_1) of the line joining any point $P(x, y)$ to the origin $(0, 0)$.
(b) Write the slope (m_2) of the line joining P to $A(0, 2)$.
(c) Find the equation of the locus of P if $m_1 - m_2 = \frac{1}{2}$.

Pages 276-277

1. 1; 45°, 135°
2. $-\frac{b}{a}$
3. a. $\frac{4}{3}$
   b. $-\frac{3}{4}$
4. Slopes are $\frac{26}{27}$ and $-\frac{27}{26}$
5. a. $M = (-\frac{5}{2}, 1)$; $N = (0, \frac{7}{2})$
   b. Slope MN = 1 = slope BC
   $MN = \frac{5}{2}\sqrt{2} = \frac{1}{2}BC$
6. a. Slope $AB = \frac{3}{4}$; slope $AC = -\frac{4}{3}$
   b. $AM = \frac{5}{2}\sqrt{5} = MB$
7. a. Each side = $\sqrt{40}$
   b. Slopes are 1 and -1
8. Slopes are $\frac{1}{a+b}$ and $-(a+b)$
9. Slopes are $\frac{b-d}{a-c}$ and $\frac{c-a}{b-d}$
10. a. $k = \frac{5}{3}$. b. $k = 15$
12. $k = 1$

Pages 278-279

1. a. $C = (a + b, c)$. b. $\left(\frac{a+b}{2}, \frac{c}{2}\right)$. c. $\left(\frac{a+b}{2}, \frac{c}{2}\right)$.

Pages 282-283

1. Slope = 4
   x-intercept = 1
   y-intercept = -4
2. Slope = $-\frac{3}{5}$
   x-intercept = 5
   y-intercept = 3
3. Slope = $\frac{1}{2}$
   x-intercept = -5
   y-intercept = $\frac{5}{2}$
4. Slope = -2
   x-intercept = -4
   y-intercept = -8
5. Slope = $-\frac{3}{2}$
   x-intercept = 2
   y-intercept = 3
6. Slope = $\frac{3}{2}$
   x-intercept = 4
   y-intercept = -6
7. $y = 3x + 4$
8. $y = -\frac{1}{2}x + 3$
9. $y = 2x - 3$
10. $y = 1.5x$
11. a. $y = 2.5x$
    b. $y = 2.5x$
13. The point (0,0)
14. $y = 63$
15. $x - y + 2 = 0$
16. $2x + y = 0$
17. $5x - 2y + 10 = 0$
18. $2x + 3y = 0$
19. $x - 2y + 10 = 0$
20. $2x - 3y - 12 = 0$
21. $y = 6$
22. $2x - y + 7 = 0$
23. $2x - y - 8 = 0$

24. $x + 3y - 18 = 0$

25. $3x - 2y + 4 = 0$

26. a. $M = (2,3)$

    b. $x - 3y + 7 = 0$

27. a. $2x + 3y - 17 = 0$

    b. $PA = PB = \frac{1}{2}\sqrt{377}$ units

28. a. $x + 2y - 5 = 0$

    b. $PA = PB = \frac{5}{2}\sqrt{5}$ units

Pages 283-284

1. Slope = 2
   x-intercept = 4
   y-intercept = -8

2. Slope = $-\frac{1}{3}$
   x-intercept = 9
   y-intercept = 3

3. Slope = $\frac{5}{2}$
   x-intercept = -4
   y-intercept = 10

4. Slope = -1
   x-intercept = 6
   y-intercept = 6

5. Slope = $\frac{2}{3}$
   x-intercept = -3
   y-intercept = 2

6. Slope = $-\frac{3}{2}$
   x-intercept = $\frac{4}{3}$
   y-intercept = 2

7. $y = 2x - 3$

8. $y = -\frac{2}{3}x + 2$

9. $y = -\frac{1}{2}x$

10. Graph (a)
    Equation of (a): $y = 2x$
    Equation of (b): $y = 2x - 6$

12. $x - y + 6 = 0$

13. $2x + 3y = 0$

14. $3x - 2y + 12 = 0$

15. $2x - y - 1 = 0$

16. $x - 2y + 4 = 0$

17. $2x + 3y - 6 = 0$

18. $2x + y = 0$

19. $y + 4 = 0$

20. $x - 5 = 0$

21. $x + 3y - 3 = 0$

22. $4x - y + 7 = 0$

23. $3x - 2y - 6 = 0$

24. $5x + 2y - 12 = 0$

25. a. $2x - y + 6 = 0$

    b. $PA = \sqrt{10} = PB$

26. a. $x - 4y - 7 = 0$

    b. $PA = \sqrt{34} = PB$

Pages 285-286

1. Intersect at (3.4,1.2)
   Perpendicular

2. Parallel

3. Intersect at (5,-1)
   Not perpendicular

4. Parallel

5. Intersect at $(0,-\frac{4}{3})$
   Not perpendicular
6. Intersect at $(-4,4)$
   Not perpendicular
7. Intersect at $(-2,1)$
   Not perpendicular
8. Intersect at $(3,-2)$
   Perpendicular
9. $4x + y - 13 = 0$
10. $x - 8y + 33 = 0$
11. Concurrent at $(10,3)$
12. a. $4x + 3y - 2 = 0$
    b. Intersection at $(\frac{4}{5},-\frac{2}{5})$; distance = 3 units
13. $3x - 4y + 6 = 0$
    $x + 4y - 6 = 0$
    Intersect at $(0,\frac{3}{2})$
14. $4x + y - 8 = 0$
    $4x - 5y + 8 = 0$
    $4x + 7y - 24 = 0$
15. a. A $(0,1)$; B $(4,0)$; C $(3,5)$
    b. $3x + 4y - 12 = 0$

Pages 286-287

1. $(\frac{3}{2},-\frac{2}{3})$. 2. $(\frac{5}{14},8\frac{3}{14})$. 3. $(2,\frac{3}{2})$. 4. $3x + 8y - 1 = 0$.

5. Concurrent at $(-16,-9)$

6. a. $3x - 2y - 17 = 0$
   b. Intersection at $(7,2)$; distance = $\sqrt{13}$ units

7. Intersection at $\left(\frac{b}{a+1}, \frac{b}{a+1}\right)$

8. $P = \left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$

10. a. $2x + 5y - 18 = 0$; $5x - 4y + 10 = 0$; $7x + y - 8 = 0$.
    b. Concurrent at $G(\frac{2}{3},\frac{10}{3})$

11. $CG = \frac{2}{3}\sqrt{29}$ units $= 2GM$

12. a. $x + 4y - 12 = 0$; $2x - 3y + 4 = 0$; $3x + y - 8 = 0$.
    b. Concurrent at $H(\frac{20}{11},\frac{28}{11})$

13. a. $x + 4y - 15 = 0$; $2x - 3y + 11 = 0$; $3x + y - 4 = 0$.
    b. Concurrent at $P(\frac{1}{11},\frac{41}{11})$

14. $PA = PB = PC = \frac{1}{11}\sqrt{2210}$ units. 15. Slope $GP = -\frac{13}{19}$ = slope PH

Pages 287-288

1. a. $PA = \sqrt{(x-4)^2 + y^2}$
      $PB = \sqrt{x^2 + (y-2)^2}$
   b. $2x - y - 3 = 0$
2. a. $2x + 3y - 17 = 0$
   b. $2ax - 2by - a^2 + b^2 = 0$
3. Concurrent at $\left(\frac{a}{2}, \frac{b^2 - ab + c^2}{2c}\right)$
4. Concurrent at $\left(\frac{a+b}{3}, \frac{c}{3}\right)$