12. $\frac{1}{3}, \frac{2}{4}, \frac{3}{5}, \frac{4}{6}, \frac{5}{7}, \frac{6}{8}$

13. $1 - \frac{1}{2}, \frac{1}{2} - \frac{1}{3}, \frac{1}{3} - \frac{1}{4}, \frac{1}{4} - \frac{1}{5}, \frac{1}{5} - \frac{1}{6}, \frac{1}{6} - \frac{1}{7}$

14. $\frac{1}{4}, \frac{2}{9}, \frac{3}{16}, \frac{4}{25}, \frac{5}{36}, \frac{6}{49}$

15. $\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \frac{5}{6}, -\frac{6}{7}$

16. 2, 6, 12, 20, 30, 42, 56

✦17. Consider the sequence defined by $a_n = \dfrac{2n + (-1)^n - 1}{4}$ for all integers $n \geq 0$. Find an alternative explicit formula for a_n that uses the floor notation.

18. Let $a_0 = 2$, $a_1 = 3$, $a_2 = -2$, $a_3 = 1$, $a_4 = 0$, $a_5 = -1$, and $a_6 = -2$. Compute each of the summations and products below.

a. $\sum_{i=0}^{6} a_i$ b. $\sum_{i=0}^{0} a_i$ c. $\sum_{j=1}^{3} a_{2j}$ d. $\prod_{k=0}^{6} a_k$ e. $\prod_{k=2}^{2} a_k$

Compute the summations and products in 19–25.

19. $\sum_{k=1}^{5} (k + 1)$

20. $\prod_{k=2}^{4} k^2$

21. $\sum_{m=0}^{4} \frac{1}{2^m}$

22. $\prod_{j=1}^{5} (-1)^j$

23. $\sum_{k=-1}^{1} (k^3 + 2)$

24. $\sum_{n=1}^{10} \left(\frac{1}{n} - \frac{1}{n+1}\right)$

25. $\prod_{i=2}^{5} \frac{(i-1) \cdot i}{(i+1) \cdot (i+2)}$

Write the summations in 26–28 in expanded form.

26. $\sum_{i=1}^{n} (-2)^i$

27. $\sum_{j=1}^{n} j(j+1)$

28. $\sum_{k=0}^{n} \frac{1}{k!}$

Write each of 29–38 using summation or product notation.

29. $1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + 7^2$

30. $(1^3 - 1) + (2^3 - 1) + (3^3 - 1) + (4^3 - 1)$

31. $(2^2 + 1) \cdot (3^2 + 1) \cdot (4^2 + 1)$

32. $\frac{2}{3 \cdot 4} + \frac{3}{4 \cdot 5} + \frac{4}{5 \cdot 6} + \frac{5}{6 \cdot 7} + \frac{6}{7 \cdot 8}$

33. $1 + r + r^2 + r^3 + r^4 + r^5$

34. $(1 - r) \cdot (1 - r^2) \cdot (1 - r^3) \cdot (1 - r^4)$

35. $1^3 + 2^3 + 3^3 + \cdots + n^3$

36. $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!}$

37. $n + (n - 1) + (n - 2) + \cdots + 1$

38. $n + \frac{n-1}{2!} + \frac{n-2}{3!} + \frac{n-3}{4!} + \cdots + \frac{1}{n!}$

Transform each of 39 and 40 by making the change of variable $i = k + 1$.

39. $\sum_{k=0}^{5} k \cdot (k - 1)$

40. $\prod_{k=1}^{n} \frac{k^2}{k+1}$

Transform each of 41–44 by making the change of variable $j = i - 1$.

41. $\sum_{i=1}^{n+1} \frac{(i-1)^2}{i}$

42. $\sum_{i=3}^{n+1} \frac{i}{i + n - 1}$

43. $\sum_{i=1}^{n-1} \frac{i}{(n-i)^2}$

44. $\prod_{i=n}^{2n} \frac{n - i + 1}{i}$

Write each of 45 and 46 as a single summation or product.

45. $3 \cdot \sum_{k=1}^{n} (2k - 3) + \sum_{k=1}^{n} (4 - 5k)$

46. $\left(\prod_{k=1}^{n} \frac{k}{k+1}\right) \cdot \left(\prod_{k=1}^{n} \frac{k+1}{k+2}\right)$

47. Check Theorem 4.1.1 for $m = 1$ and $n = 4$ by writing out the left- and right-hand sides of the equations in expanded form. The two sides are equal by repeated application of certain laws. What are these laws?

Compute each of 48–56.

48. $\frac{4!}{3!}$

49. $\frac{5!}{7!}$

50. $\frac{6!}{0!}$

51. $\frac{n!}{(n-1)!}$

52. $\frac{(n-1)!}{(n+1)!}$

53. $\frac{n!}{(n-2)!}$

54. $\frac{((n+1)!)^2}{(n!)^2}$

55. $\frac{n!}{(n-k)!}$

56. $\frac{n!}{(n-k-1)!}$

57. a. Prove that $n! + 2$ is divisible by 2, for all integers $n \geq 2$.

b. Prove that $n! + k$ is divisible k, for all integers $n \geq 2$ and $k = 2, 3, \ldots, n$.

H c. Given any integer $m \geq 2$, is it possible to find a sequence of $m - 1$ consecutive positive integers none of which is prime? Explain your answer.

58. Suppose $a[1], a[2], a[3], \ldots, a[m]$ is a one-dimensional array and consider the following algorithm segment:

```
sum := 0
for k := 1 to m
    sum := sum + a[k]
next k
```

21. a. *Hint 1:* If $a = dq - r$, then $-a = -dq - r = -dq - d + d - r = d(-q - 1) + (d - r)$.

Hint 2: If $0 \le r < d$, then $0 \ge -r > -d$. Add d to all parts of this inequality and see what results.

24. a. $\text{lcm}(12, 18) = 36$

25. *Proof:* Let a and b be positive integers and suppose $d = \gcd(a, b) = \text{lcm}(a, b)$. By definition of greatest common divisor and least common multiple, $d > 0$, $d \mid a$, $d \mid b$, $a \mid d$, and $b \mid d$. Thus, in particular, $a = d \cdot m$ and $d = a \cdot n$ for some integers m and n. By substitution, $a = d \cdot m = (a \cdot n) \cdot m = a \cdot nm$. Dividing both sides by a gives $1 = nm$. But the only divisors of 1 are 1 and -1 (Example 3.3.4), and so $m = n = \pm 1$. Since both a and d are positive, $m = n = 1$, and hence $a = d$. Similar reasoning show that $b = d$ also, and so $a = b$.

SECTION 4.1

1. $\frac{1}{9}, \frac{2}{8}, \frac{3}{7}, \frac{4}{6}$

3. $1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}$

5. 0, 0, 2, 2

8. $g_1 = \lfloor \log_2 1 \rfloor = 0$
$g_2 = \lfloor \log_2 2 \rfloor = 1, \quad g_3 = \lfloor \log_2 3 \rfloor = 1$
$g_4 = \lfloor \log_2 4 \rfloor = 2, \quad g_5 = \lfloor \log_2 5 \rfloor = 2$
$g_6 = \lfloor \log_2 6 \rfloor = 2, \quad g_7 = \lfloor \log_2 7 \rfloor = 2$
$g_8 = \lfloor \log_2 8 \rfloor = 3, \quad g_9 = \lfloor \log_2 9 \rfloor = 3$
$g_{10} = \lfloor \log_2 10 \rfloor = 3, \quad g_{11} = \lfloor \log_2 11 \rfloor = 3$
$g_{12} = \lfloor \log_2 12 \rfloor = 3, \quad g_{13} = \lfloor \log_2 13 \rfloor = 3$
$g_{14} = \lfloor \log_2 14 \rfloor = 3, \quad g_{15} = \lfloor \log_2 15 \rfloor = 3$

When n is an integral power of 2, g_n is the exponent of that power. For instance, $8 = 2^3$ and $g_8 = 3$. More generally, if $n = 2^k$, where k is an integer, then $g_n = k$. All terms of the sequence from g_n up to g_m, where $m = 2^{k+1}$ is the next integral power of 2, have the same value as g_n, namely k. For instance, all terms of the sequence from g_8 through g_{15} have the value 3.

Exercises 10–16 have more than one correct answer.

10. $a_n = (-1)^n$, where n is an integer and $n \ge 1$.

11. $b_n = (n - 1) \cdot (-1)_n$, where n is an integer and $n \ge 1$.

12. $c_n = \dfrac{n}{n + 2}$, where n is an integer and $n \ge 1$.

18. a. $2 + 3 + (-2) + 1 + 0 + (-1) + (-2) = 1$
b. $a_0 = 2$
c. $a_2 + a_4 + a_6 = -2 + 0 + (-2) = -4$
d. $2 \cdot 3 \cdot (-2) \cdot 1 \cdot 0 \cdot (-1) \cdot (-2) = 0$

19. $2 + 3 + 4 + 5 + 6 = 20$

20. $2^2 \cdot 3^2 \cdot 4^2 = 576$

24. $$\left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \left(\frac{1}{6} - \frac{1}{7}\right) + \left(\frac{1}{7} - \frac{1}{8}\right) + \left(\frac{1}{8} - \frac{1}{9}\right) + \left(\frac{1}{9} - \frac{1}{10}\right) + \left(\frac{1}{10} - \frac{1}{11}\right) = 1 - \frac{1}{11} = \frac{10}{11}$$

26. $$(-2)^1 + (-2)^2 + (-2)^3 + \cdots + (-2)^n = -2 + 2^2 - 2^3 + \cdots + (-1)^n \cdot 2^n$$

Exercises 29–38 have more than one correct answer.

29. $\displaystyle\sum_{k=1}^{7} (-1)^{k+1} \cdot k^2$ or $\displaystyle\sum_{k=0}^{6} (-1)^k \cdot (k + 1)^2$

32. $\displaystyle\sum_{j=2}^{6} \frac{j}{(j + 1) \cdot (j + 2)}$ or $\displaystyle\sum_{j=3}^{7} \frac{j - 1}{j \cdot (j + 1)}$

33. $\displaystyle\sum_{i=0}^{5} r^i$

35. $\displaystyle\sum_{k=1}^{n} k^3$

37. $\displaystyle\sum_{i=0}^{n-1} (n - i)$

39. When $k = 0$, then $i = 1$. When $k = 5$, then $i = 6$. Since $i = k + 1$, then $k = i - 1$. Thus,
$$\sum_{k=0}^{5} (k - 1) \cdot k = \sum_{i=1}^{6} (i - 2) \cdot (i - 1).$$

41. When $i = 1$, then $j = 0$. When $i = n + 1$, then $j = n$. Since $j = i - 1$, then $i = j + 1$. Thus,
$$\sum_{i=1}^{n+1} \frac{(i - 1)^2}{i} = \sum_{j=0}^{n} \frac{j^2}{j + 1}.$$

42. When $i = 3$, then $j = 2$. When $i = n + 1$, then $j = n$. Since $j = i - 1$, then $i = j + 1$. Note that n is constant as far as the sum is concerned. Thus,
$$\sum_{i=3}^{n+1} \frac{i}{i + n - 1} = \sum_{j=2}^{n} \frac{j + 1}{(j + 1) + n - 1} = \sum_{j=2}^{n} \frac{j + 1}{j + n}.$$

45. $$\sum_{k=1}^{n} [3(2k - 3) + (4 - 5k)] = \sum_{k=1}^{n} [(6k - 9) + (4 - 5k)] = \sum_{k=1}^{n} (k - 5)$$

48. $$\frac{4 \cdot \cancel{3 \cdot 2 \cdot 1}}{\cancel{3 \cdot 2 \cdot 1}} = 4$$

51. $$\frac{n \cdot \cancel{(n - 1) \cdot (n - 2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1}}{\cancel{(n - 1) \cdot (n - 2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1}} = n$$

52. $$\frac{\cancel{(n-1)\cdot(n-2)\cdot\ldots\cdot 3\cdot 2\cdot 1}}{(n+1)\cdot n\cdot \cancel{(n-1)\cdot(n-2)\cdot\ldots\cdot 3\cdot 2\cdot 1}} = \frac{1}{n(n+1)}$$

54. $$\frac{[(n+1)\cdot n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot 3\cdot 2\cdot 1]^2}{(n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot 3\cdot 2\cdot 1)^2} = (n+1)^2$$

55. $$\frac{n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot(n-k+1)\cdot\cancel{(n-k)\cdot(n-k-1)\cdot\ldots\cdot 2\cdot 1}}{\cancel{(n-k)\cdot(n-k-1)\cdot\ldots\cdot 2\cdot 1}} = n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot(n-k+1)$$

57. a. *Proof*: Let n be an integer such that $n \geq 2$. By definition of factorial,

$$n! = \begin{cases} 2\cdot 1 & \text{if } n = 2 \\ 3\cdot 2\cdot 1 & \text{if } n = 3 \\ n\cdot(n-1)\cdot\ldots\cdot 2\cdot 1 & \text{if } n > 3. \end{cases}$$

In each case, $n!$ has a factor of 2, and so

$n! = 2\cdot k$ for some integer k.

Then

$$\begin{aligned} n! + 2 &= 2\cdot k + 2 && \text{by substitution} \\ &= 2\cdot(k+1) && \text{by factoring out the 2.} \end{aligned}$$

Since $k + 1$ is an integer, $n! + 2$ is divisible by 2 *[as was to be shown]*.

c. *Hint:* Consider the sequence $m! + 2$, $m! + 3$, $m! + 4, \ldots, m! + m$.

58. a. $m - 1$, $sum + a[i + 1]$

59.

$$\begin{array}{r|ll} 2 & 90 & \\ 2 & 45 & \text{remainder} = r[0] = 0 \\ 2 & 22 & \text{remainder} = r[1] = 1 \\ 2 & 11 & \text{remainder} = r[2] = 0 \\ 2 & 5 & \text{remainder} = r[3] = 1 \\ 2 & 2 & \text{remainder} = r[4] = 1 \\ 2 & 1 & \text{remainder} = r[5] = 0 \\ & 0 & \text{remainder} = r[6] = 1 \end{array}$$

Hence $90_{10} = 1011010_2$.

62.

a	23					
i	0	1	2	3	4	5
q	23	11	5	2	1	0
$r[0]$		1				
$r[1]$			1			
$r[2]$				1		
$r[3]$					0	
$r[4]$						1

66.

$$\begin{array}{r|ll} 16 & 287 & \\ 16 & 17 & \text{remainder } 15 = r[0] = F_{16} \\ 16 & 1 & \text{remainder } 1 = r[1] = 1_{16} \\ & 0 & \text{remainder } 1 = r[2] = 1_{16} \end{array}$$

Hence $287_{10} = 11F_{16}$.

SECTION 4.2

1. Let $P(n)$ be the property "n cents can be obtained by using 3-cent and 8-cent coins."

The property is true for $n = 15$:

Fifteen cents can be obtained by using five 3-cent coins.

If the property is true for $n = k$ then it is true for $n = k + 1$:

Suppose k cents (where $k \geq 15$) can be obtained using 3-cent and 8-cent coins. *[Inductive hypothesis]* We must show that $k + 1$ cents can be obtained using 3-cent and 8-cent coins. If the k cents includes an 8-cent coin, replace it by three 3-cent coins to obtain a total of $k + 1$ cents. If the k cents consists of 3-cent coins exclusively, then there must be at least five 3-cent coins (since the total amount is at least 15 cents). In this case, replace five of the 3-cent coins by two 8-cent coins to obtain a total of $k + 1$ cents. Thus, in either case, $k + 1$ cents can be obtained using 3-cent and 8-cent coins. *[This is what we needed to show.]*

[Since we have proved the basis step and the inductive step, we conclude that the given statement is true for all integers $n \geq 15$.]

3. a. $P(1)$ is "$1^2 = \dfrac{1\cdot(1+1)\cdot(2\cdot 1+1)}{6}$." $P(1)$ is true because $1^2 = 1$ and $\dfrac{1\cdot(1+1)\cdot(2+1)}{6} = \dfrac{2\cdot 3}{6} = 1$ also.

b. $P(k)$ is "$1^2 + 2^2 + \cdots + k^2 = \dfrac{k(k+1)(2k+1)}{6}$