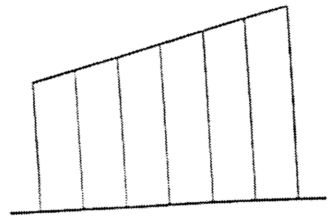
Exercises [A-1]

1. An arithmetic series has $t_1 = 4$, $t_2 = 7$. Find t_{10}, S_{10}.
2. An arithmetic series has $t_1 = 6$ and common difference 2. Find t_{12}, S_{12}.
3. An arithmetic series has $t_1 = 5$, $t_2 = 8$. Find t_n, S_n.
4. Find the first term, the common difference, and t_n for a series having $S_n = 2n^2 - 7n$.
5. Find the first term, the common difference, and t_n for a series having $S_n = 4n^2 + n$.
6. Find the first term, the common difference, and t_n for a series having $S_n = 4n - 3n^2$.
7. If $t_n = 2n - 1$, find S_n and evaluate S_{20}.
8. If $t_n = n$, find S_n and evaluate S_{100}.
9. Find S_{10} for the arithmetic series $5 + 8 + 11 + \cdots$.
10. Find S_{15} for the arithmetic series $3 + 13 + 23 + \cdots$.
11. Find S_{20} for the arithmetic series $13 + 8 + 3 + \cdots$.
12. Find S_{12} for the arithmetic series $-8 - 2 + 4 + \cdots$.
13. Find S_{10} for the arithmetic series $16 + 13.6 + 11.2 + \cdots$.
14. Find S_{20} for the arithmetic series $-2.8 - 4.0 - 5.2 - \cdots$.
15. Find the sum of the positive even integers up to and including 100.
16. Find the sum of the arithmetic series $5 + 4.8 + 4.6 + \cdots + 1.2 + 1.0$.
17. (a) Find the sum of all positive integers less than 100 which are divisible by 3.
    (b) Find the sum of all positive integers less than 100 which are not divisible by 3.
18. (a) Find the sum of all positive integers less than 200 which are divisible by 5.
    (b) Find the sum of all positive odd integers less than 200 which are divisible by 5.
19. In an arithmetic series $t_8 = 24$, $t_{12} = 30$. Find S_{12}.
20. In an arithmetic series $t_4 = 4$, $t_8 = -20$. Find S_{10}.
21. Find S_n for the arithmetic series $4 + 7 + 10 + \cdots$ and determine the value of n for which the series has sum 175.
22. Find S_n for the arithmetic series $14 + 10 + 6 + \cdots$ and determine the value of n for which $S_n = -18$.
23. In an arithmetic series $t_1 = 2.5$, $t_3 = 2$. Find the value of n for which $S_n = 13.5$.

24. In an arithmetic series $S_{12} = 21$ and $t_{12} = 10$. Find t_1 and t_2.

25. An arithmetic series contains 20 terms. Show that if $t_1 = a - b$ and $t_{20} = a + b$, the value of S_{20} is independent of b.

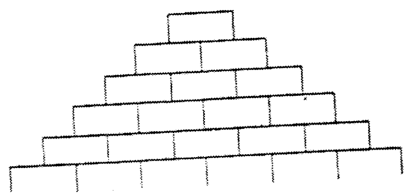
26. The side of a shed is 9 ft. high at the front, 6 ft. high at the back, and 12 ft. from back to front. If supporting poles are placed at 2-foot intervals from front to back, find the total length of these supports.



27. The rungs of a ladder decrease from a width of 2 ft. 4 in. at the bottom to a width of 1 ft. 8 in. at the top. If there are 25 rungs, find (a) the difference in length from one rung to the next, (b) the total length of the 25 rungs.

28. A boy was given an allowance of 25 cents a week beginning on his sixth birthday. On each birthday following this the weekly allowance was increased 15 cents. (a) What is the weekly allowance for the year beginning on his 15th birthday? (b) Taking 52 weeks in each year, find the total amount of his allowance between his sixth and sixteenth birthdays.

29. A gable end is built with 24 bricks in the lowest row, and one less brick in each row than in the row beneath. Find the total number of bricks required.

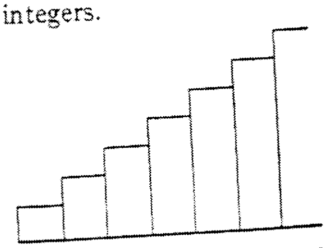


30. (a) Show that the sum of the first n positive integers is $\frac{n(n+1)}{2}$.

(b) Show how the sum of the first n positive even integers can be deduced from the result obtained in (a).

(c) Deduce expressions for the sum of the first $2n$ positive integers and the sum of the first n positive odd integers.

31. The lowest step of a staircase is 8 in. high, and the others each rise 6 in. above the one below. If there are 15 steps, find the total length of board required to enclose the two sides of the staircase.



32. A wire rope is wound on a drum. The first 10 turns are each of length 12 ft., the next 10 turns are each of length 12.4 ft., the next 10 of length 12.8 ft., and so on. Find the length of rope in 100 turns.

Exercises [A-2]

1. Find t_{10} in an arithmetic progression having $t_4 = 10$ and $t_9 = 4$.
2. Find the sum of the first 15 terms of an arithmetic series if the first term is 6 and the fifteenth term is -15.
3. Find the sum of 20 terms of the arithmetic series $12 + 7 + \cdots$.
4. Find the first three terms of an arithmetic progression in which $t_5 = 6$ and $t_9 = 0$.
5. Find the fifteenth term of the arithmetic sequence $x, \frac{1}{2}(x + y), \cdots$.
6. Find the sum of the odd integers from 1 through 101.
7. Find t_{10} and S_{10} for the arithmetic series in which $t_5 = 8.6$ and $t_8 = 12.2$.
8. The first term of an arithmetic series is 4 and the sum of the first 16 terms is 280. Find the common difference.
9. Find the sum of the numbers $3 + 7 + 11 + \cdots + 59 + 63$.
10. Find the first term and the common difference of an arithmetic progression in which $t_4 = 3x - y$ and $t_{16} = 7y - x$.
11. Find the first three terms of an arithmetic series in which the tenth term is -3 and the sum of the first ten terms is $37\frac{1}{2}$.
12. Write the first three terms of the series for which $t_n = 1.5(n + 1)$. Find the number of terms of the series required to make the sum 135.
13. Is -21 a term of the arithmetic sequence having $t_1 = 90$, $t_3 = 81$?
14. If the first and last terms of an arithmetic series are 5 and 25, show that the sum of the series varies directly as the number of terms.
15. Find the sum of all two-digit numbers ending in 2 or 8.
16. Find the sum of 24 terms of an arithmetic series of which the first term is a and the tenth term is $7a$.
17. How many terms of the series $25 + 19 + 13 + \cdots$ are required to make the sum -20?
18. If $t_n = 4n + 2$, show that $S_n = 2n^2 + 4n$. Find n if $S_n = 510$.
19. A ball rolling down an inclined plane travels, in successive seconds, k feet, $2.5k$ feet, $4k$ feet, $\cdots$. Find the distance traveled by the ball (**a**) in the ninth second, (**b**) in the first 15 seconds.
20. A machine for cutting lengths of metal rod is such that when the pointer on a dial is set at x, each piece cut is x inches longer than the one before. A mechanic needs 36 such pieces ranging from 24 in. to 38 in. Find the required setting on the dial, and the total length of the 36 pieces.

It is of interest to find the difference between 8 and S_n for a large value of n.

We have $$8 - S_n = 8 - 8 + 8(\tfrac{1}{2})^n = 8(\tfrac{1}{2})^n = 2^3 \cdot \frac{1}{2^n}$$
$$= \frac{1}{2^{n-3}}$$

For convenience, let us take $n = 103$.

Then $$8 - S_n = \frac{1}{2^{100}}$$
$$= 10^{-30} \quad \text{(approximately)}$$

Thus the difference between 8 and the sum of 103 terms of the series
$$4 + 2 + 1 + \tfrac{1}{2} + \ldots$$
is a number which is approximately 0.000,000,000,000,000,000,000,000,000,001.

Exercises [A-1]

1. Find t_{10} and S_{10} for the geometric series $1 + 2 + \cdots$.
2. Find t_8 and S_8 for the geometric series $8 + 4 + \cdots$.
3. Find t_8 and S_8 for the geometric series $8 - 4 + \cdots$.
4. Find S_9 for the geometric series $9 - 3 + \cdots$.
5. In a geometric series $t_2 = 6$, $t_5 = 162$. Find S_8.
6. Find t_{10} and S_{10} for the geometric series $10 + 20 + \cdots$.
7. Use synthetic division to show that $\dfrac{x^5 - 1}{x - 1} = x^4 + x^3 + x^2 + x + 1$, $(x \neq 1)$.
8. Given that $(\frac{1}{2})^{10} = 0.00098$ (nearly), find to 3 decimal places the value of S_{10} for the geometric series $1 + \frac{1}{2} + \frac{1}{4} + \cdots$.
9. Given that $(\frac{2}{3})^{12} = 0.0077$ (nearly), find to 3 decimal places the value of S_{12} for the geometric series $1 + \frac{2}{3} + \frac{4}{9} + \cdots$.
10. Show that $1 + 2 + 4 + \cdots + 2^{n-2} + 2^{n-1} = 2^n - 1$.
11. Show that $1 + 3 + 9 + \cdots + 3^{n-2} + 3^{n-1} = \frac{1}{2}(3^n - 1)$.
12. Show that $1 + \dfrac{1}{2} + \dfrac{1}{4} + \cdots + \dfrac{1}{2^{n-2}} + \dfrac{1}{2^{n-1}} = 2 - \dfrac{1}{2^{n-1}}$.
13. Show that if a geometric series has $r = 2$, then $S_n = t_{n+1} - t_1$.
14. Show that for any geometric series with common ratio r,
$$rS_n = S_{n+1} - t_1.$$
By subtracting S_n from each side, show that
$$S_n = \frac{t_{n+1} - t_1}{r - 1}.$$

15. Find to 3 significant figures the value of S_{10} for the geometric series $3 + 4.5 + 6.75 + \cdots$.
16. Find to 3 significant figures the value of S_{12} for the geometric series $8 - 4 + 2 - \cdots$.
17. Find to 3 significant figures the value of S_9 for the geometric series $6 + 7.2 + 8.64 + \cdots$.
18. In a geometric series $t_1 = 4.5$, $t_2 = 3.6$. Find to 3 significant figures the value of S_{15}.
19. If in a geometric series $t_1 = 5$ and $t_4 = 10$, use logarithms to find r and S_{10} for the series.
20. A chessboard contains 64 squares. A cent is placed on the first square, 2 cents on the second, 4 cents on the third, and so on, the sums of money forming a geometric sequence. If all the squares are filled in this way, find the total value of the money.

Exercises [A-2]

1. Show that 3.375, 2.25, 1.5 form a geometric sequence.
2. Find the third term of the geometric sequence 4.8, 3.6, $\cdots$.
3. Find x if x, $x + 3$, $x + 12$ are in geometric progression.
4. Find the eighth term of the geometric sequence $9, -6, \cdots$.
5. Find the common ratio of a geometric progression in which $t_4 = 5$ and $t_8 = 80$.
6. Find the sum of the first ten terms of the series $8 + 4 + 2 + \cdots$.
7. Find the first term of a geometric series in which $t_4 = \frac{3}{4}$, $t_6 = \frac{1}{3}$.
8. Find to three significant figures the tenth term of the sequence 5, 6, 7.2, $\cdots$.
9. Find the sixth term of the series $\frac{4}{3} - 4 + 12 - \cdots$.
10. Find the sum of $3^1 + 3^2 + 3^3 + \cdots + 3^{10}$, given $3^{10} = 59{,}049$.
11. Find S_{10} for the series $4.8 + 3.6 + \cdots$, (a) if it is arithmetic, (b) if it is geometric.
12. What is the third term of a geometric sequence with first two terms a, b?
13. Find the common ratio of a geometric sequence in which the sum of the third and fourth terms is 4 times the sum of the first and second.
14. Find to 3 significant figures the sum of 12 terms of the series $20 - 12 + 7.2 - \cdots$.

15. A geometric series has $t_1 = 10$ and $t_4 = 20$. Find the common ratio and the sum of the first nine terms, each to 3 significant figures.

16. Determine whether three successive terms of an arithmetic sequence can also be successive terms of a geometric sequence.

17. A car depreciates in such a way that its value at the end of a year is 70% of its value at the start of the year. Find its value at the end of 5 years if it originally was valued at \$2500.

18. Each stroke of an air pump reduces the amount of air in a container by 25%. What percentage of the air originally in the container remains in it after 8 strokes of the pump?

19. If the sum of \$$X$ is invested at 4% compound interest, what is the amount of the investment at the end of 1 year; 2 years; n years?

20. (a) If \$1000 is invested at 4% compound interest, what is the value of the investment at the end of n years?
    (b) A man invested \$1000 at 4% compound interest on the first of January in each of the years 1955 through 1964. Find the total value of the investments on December 31, 1964.

Limit of the Sum of a Geometric Series

In obtaining the sum formulas for the arithmetic and the geometric series we have considered a series to have a definite, though not necessarily specified, number of terms. A wide and important field in mathematics is the study of series in which the number of terms is unlimited. Expressions such as

$$\pi = 4 - \tfrac{4}{3} + \tfrac{4}{5} - \tfrac{4}{7} + \tfrac{4}{9} - \tfrac{4}{11} + \cdots,$$

and

$$e = 1 + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} + \cdots,$$

are examples of such series.

To see how such a series may arise in even an elementary situation, consider the problem of representing the fraction $\frac{1}{3}$ in the decimal notation. We have

$$\tfrac{1}{3} = \tfrac{3}{10}, \quad \text{or } 0.3, \quad \text{with remainder } \tfrac{1}{30},$$

$$\tfrac{1}{3} = \tfrac{3}{10} + \tfrac{3}{100}, \quad \text{or } 0.33, \quad \text{with remainder } \tfrac{1}{300},$$

$$\tfrac{1}{3} = \tfrac{3}{10} + \tfrac{3}{100} + \tfrac{3}{1000}, \quad \text{or } 0.333, \quad \text{with remainder } \tfrac{1}{3000},$$

and so on. More and more terms may be formed in the series

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \cdots + \frac{3}{10^n} + \cdots, \qquad ①$$

but at no stage is the transformation of $\frac{1}{3}$ into the decimal notation complete.

17. 10

18. $n = 15$

19. a. $13k$ ft.

b. $\frac{345k}{2}$ ft.

20. $x = 0.4$; 93 ft.

Pages 503-504

1. $r = 3$

18, 54, 162

2. $r = \frac{1}{3}$

$\frac{2}{3}, \frac{2}{9}, \frac{2}{27}$

3. $r = -3$

18, -54, 162

4. 640

5. $-\frac{1}{128}$

6. 8.4×10^{-7}

7. -0.0026

8. 2916

9. 1004

10. 10.0001

11. $t_1 = \frac{1}{8}$; $r = 2$ or -2

12. $r = 1.10$

13. New value $= k(1 + \frac{x}{100})^2$

Final value $= k(1 + \frac{x}{100})^n$

15. 61%

16. a. \$D(1.05), \$D(1.10), \$D(1.15), \$D(1.20), \$D$(1 + \frac{n}{20})$

b. \$D(1.05), \$D$(1.05)^2$, \$D$(1.05)^3$, \$D$(1.05)^4$, \$D$(1.05)^n$

17. a. \$500. b. \$629

19. $\frac{5}{64}$ gm.

20. 1, 2, 2^2, 2^3, $\cdots$, 2^n; $2^{1,000,000}$; 300,000

Pages 506-507

1. $t_{10} = 512$; $S_{10} = 1023$

2. $t_8 = \frac{1}{16}$; $S_8 = 15\frac{15}{16}$

3. $t_8 = -\frac{1}{16}$; $S_8 = 5\frac{5}{16}$

4. $6\frac{547}{729}$

5. 6560

6. $t_{10} = 5120$; $S_{10} = 10{,}230$

8. 1.998

9. 2.977

15. 340

16. 5.33

17. 125

18. 21.7

19. $r = 1.260$; $S_{10} = 174.6$

20. (1.836×10^{17}) dollars

Pages 507-508

2. 2.7

3. $x = \frac{3}{2}$

4. $-\frac{128}{243}$

5. $r = 2$ or -2

6. $15\frac{63}{64}$

7. $\frac{81}{32}$ or $-\frac{81}{32}$

8. 25.8

9. -324

10. 88,572

11. a. -6.0

b. 18.1

12. $\frac{b^2}{a}$

13. $r = 2$ or -2

14. 12.5

15. $r = 1.26$

$S_9 = 269$

16. Not generally

17. \$420

18. 10%

19. \$X(1.04)

\$X$(1.04)^2$

\$X$(1.04)^n$

20. a. \$1000$(1.04)^n$

b. \$12,450, approx.

Pages 511-513