[S11-05-09-15] Inverse Functions

3.2 INVERSE FUNCTIONS

The common theme that links the functions of this chapter is that they occur as pairs of inverse functions. The only functions that possess inverse functions are one-to-one functions, so we start by reviewing that concept.

Let us compare the functions f and g whose arrow diagrams are shown in Figure 1.

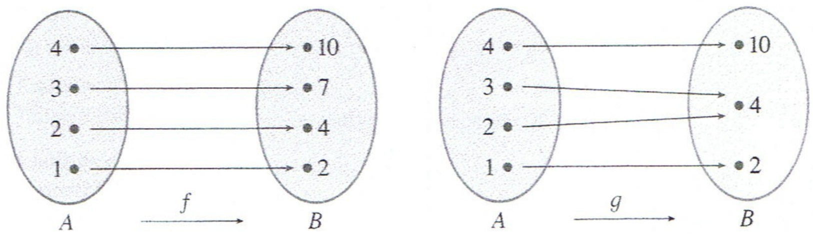


FIGURE 1

Note that f never takes on the same value twice (any two numbers in A have different images), whereas g does take on the same value twice (both 2 and 3 have the same image, 4). In symbols,

$$g(2) = g(3)$$

but $$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2$$

Functions that have this latter property are called *one-to-one functions.*

> **(1) DEFINITION** A function f with domain A is called a **one-to-one function** if no two elements of A have the same image; that is,
>
> $$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2$$

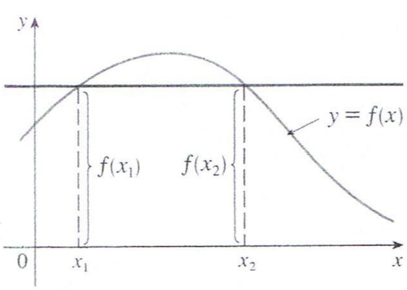


FIGURE 2
This function is not one-to-one because $f(x_1) = f(x_2)$

If a horizontal line intersects the graph of f in more than one point, then we see from Figure 2 that there are numbers x_1 and x_2 such that $f(x_1) = f(x_2)$. This means that f is not one-to-one. Therefore we have the following geometric method for determining whether a function is one-to-one.

> **HORIZONTAL LINE TEST** A function is one-to-one if and only if no horizontal line intersects its graph more than once.

EXAMPLE 1 Is the function $f(x) = x^3$ one-to-one?

SOLUTION 1 If $x_1 \neq x_2$, then $x_1^3 \neq x_2^3$ (two different numbers cannot have the same cube). Therefore, by Definition 1, $f(x) = x^3$ is one-to-one.

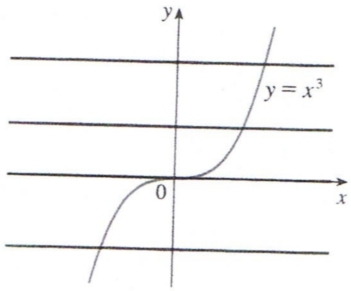


FIGURE 3
$f(x) = x^3$ is one-to-one

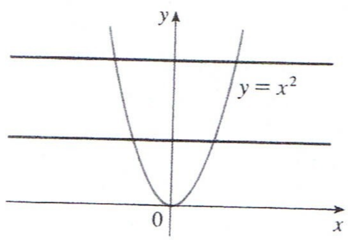


FIGURE 4
$g(x) = x^2$ is not one-to-one

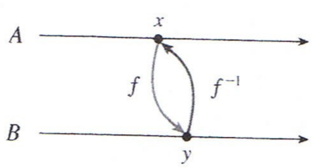


FIGURE 5

SOLUTION 2 From Figure 3 we see that no horizontal line intersects the graph of $f(x) = x^3$ more than once. Therefore, by the Horizontal Line Test, f is one-to-one. ■

EXAMPLE 2 Is the function $g(x) = x^2$ one-to-one?

SOLUTION 1 This function is not one-to-one because, for instance,

$$g(1) = 1 = g(-1)$$

and so 1 and -1 have the same image.

SOLUTION 2 From Figure 4 we see that there are horizontal lines that intersect the graph of g more than once. Therefore, by the Horizontal Line Test, g is not one-to-one. ■

One-to-one functions are important because they are precisely the functions that possess inverse functions according to the following definition.

(2) DEFINITION Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \iff f(x) = y$$

for any y in B.

This definition says that if f maps x into y, then f^{-1} maps y back into x. (If f were not one-to-one, then f^{-1} would not be uniquely defined.) The arrow diagram in Figure 5 indicates that f^{-1} reverses the effect of f. Note that

domain of f^{-1} = range of f

range of f^{-1} = domain of f

For example, the inverse function of $f(x) = x^3$ is $f^{-1}(x) = x^{1/3}$ because if $y = x^3$, then

$$f^{-1}(y) = f^{-1}(x^3) = (x^3)^{1/3} = x$$

⊘ *Caution:* Do not mistake the -1 in f^{-1} for an exponent. Thus

$$f^{-1}(x) \quad \text{does } \textit{not} \text{ mean} \quad \frac{1}{f(x)}$$

The reciprocal $1/f(x)$ could, however, be written as $[f(x)]^{-1}$.

The letter x is traditionally used as the independent variable, so when we concentrate on f^{-1} rather than on f, we usually reverse the roles of x and y in Definition 2 and write

(3) $$f^{-1}(x) = y \iff f(y) = x$$

By substituting for y in Definition 2 and substituting for x in (3), we get the following **cancellation equations:**

(4)
$$f^{-1}(f(x)) = x \quad \text{for every } x \text{ in } A$$
$$f(f^{-1}(x)) = x \quad \text{for every } x \text{ in } B$$

The first cancellation equation says that if we start with x, apply f, and then apply f^{-1}, we arrive back at x, where we started. Thus f^{-1} undoes what f does. The second equation says that f undoes what f^{-1} does.

For example, if $f(x) = x^3$, then $f^{-1}(x) = x^{1/3}$ and the cancellation equations become

$$f^{-1}(f(x)) = (x^3)^{1/3} = x$$
$$f(f^{-1}(x)) = (x^{1/3})^3 = x$$

These equations simply say that the cube function and the cube root function cancel each other.

Let us now see how to compute inverse functions. If we have a function $y = f(x)$ and are able to solve this equation for x in terms of y, then according to Definition 2 we must have $x = f^{-1}(y)$. If we then interchange x and y, we have $y = f^{-1}(x)$, which is the desired equation.

(5) HOW TO FIND THE INVERSE FUNCTION OF A ONE-TO-ONE FUNCTION f

Step 1. Write $y = f(x)$.
Step 2. Solve this equation for x in terms of y (if possible).
Step 3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.

EXAMPLE 3 Find the inverse function of $f(x) = x^3 + 2$.

SOLUTION According to (5) we first write

$$y = x^3 + 2$$

Then we solve this equation for x:

$$x^3 = y - 2$$
$$x = \sqrt[3]{y - 2}$$

Finally, we interchange x and y:

$$y = \sqrt[3]{x - 2}$$

Therefore the inverse function is $f^{-1}(x) = \sqrt[3]{x - 2}$. ∎

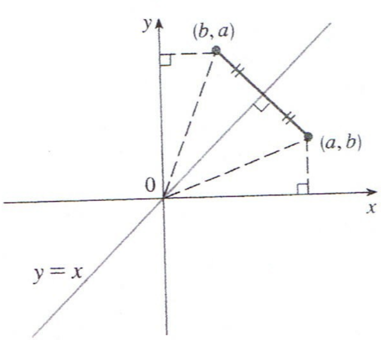


FIGURE 6

The principle of interchanging x and y to find the inverse function also gives us the method for obtaining the graph of f^{-1} from the graph of f. Since $f(a) = b$ if and only if $f^{-1}(b) = a$, the point (a, b) is on the graph of f if and only if the point (b, a) is on the graph of f^{-1}. But we get the point (b, a) from (a, b) by reflecting about the line $y = x$ (see Figure 6).

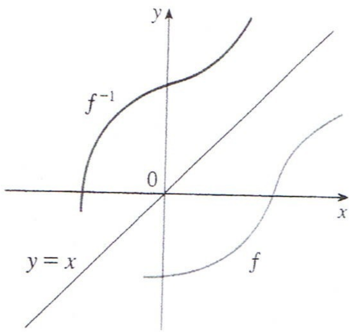


FIGURE 7

Therefore, as illustrated by Figure 7:

> The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

EXAMPLE 4 Sketch the graphs of $f(x) = \sqrt{-1 - x}$ and its inverse function using the same coordinate axes.

SOLUTION First we sketch the curve $y = \sqrt{-1 - x}$ (the top half of the parabola $y^2 = -1 - x$ or $x = -y^2 - 1$) and then we reflect about the line $y = x$ to get the graph of f^{-1} (see Figure 8).

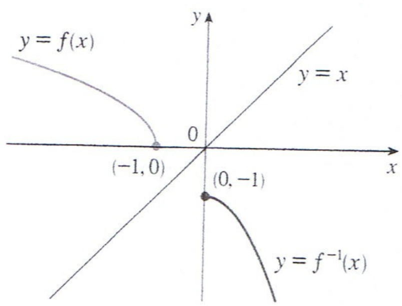


FIGURE 8

Now let us look at inverse functions from the point of view of calculus. Suppose that f is both one-to-one and continuous. We think of a continuous function as one whose graph has no break in it. (It consists of just one piece.) Since the graph of f^{-1} is obtained from the graph of f by reflecting about the line $y = x$, the graph of f^{-1} has no break in it either (see Figure 7). Thus we would expect that f^{-1} is also a continuous function.

This geometrical argument does not prove the following theorem but at least it makes the theorem plausible. A proof can be found in Appendix F.

> **(6) THEOREM** If f is a one-to-one continuous function defined on an interval, then its inverse function f^{-1} is also continuous.

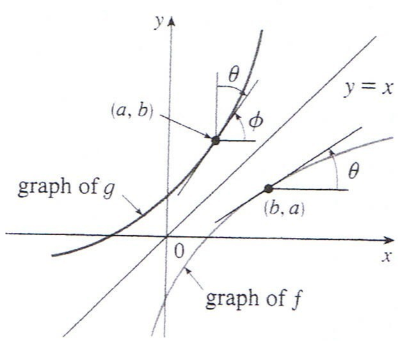


FIGURE 9

Now suppose that f is a one-to-one differentiable function. Geometrically we can think of a differentiable function as one whose graph has no corner or kink in it. We get the graph of f^{-1} by reflecting the graph of f about the line $y = x$, so the graph of f^{-1} has no corner or kink in it either. We therefore expect that f^{-1} is also differentiable (except where its tangents are vertical). In fact, we can predict the value of the derivative of f^{-1} at a given point by a geometric argument. In Figure 9 the graphs of f and its inverse $g = f^{-1}$ are shown. If $f(b) = a$, then $g(a) = f^{-1}(a) = b$ and $g'(a)$ is the slope of the tangent to the graph of g at (a, b), which is $\tan \phi$. Likewise, $f'(b) = \tan \theta$. From Figure 9 we see that $\theta + \phi = \pi/2$, so

$$g'(a) = \tan \phi = \tan\left(\frac{\pi}{2} - \theta\right) = \frac{1}{\tan \theta} = \frac{1}{f'(b)}$$

that is,

$$g'(a) = \frac{1}{f'(g(a))}$$

> **(7) THEOREM** If f is a one-to-one differentiable function with inverse function $g = f^{-1}$ and $f'(g(a)) \neq 0$, then the inverse function is differentiable at a and
>
> $$g'(a) = \frac{1}{f'(g(a))}$$

PROOF Write the definition of derivative as in Equation 2.1.3:

$$g'(a) = \lim_{x \to a} \frac{g(x) - g(a)}{x - a}$$

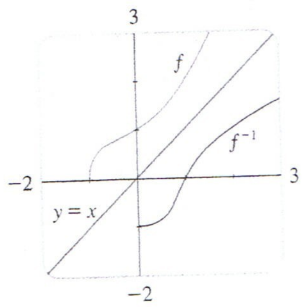


FIGURE 14

Let's also plot the line $y = x$:

$$x = t \qquad y = t$$

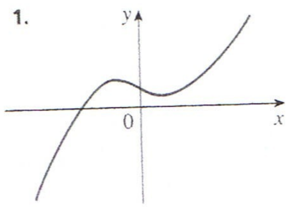
Figure 14 shows all three graphs and, indeed, it appears that the graph of f^{-1} is the reflection of the graph of f in the line $y = x$. ■

In Exercise 38 we show how to graph the derivative of the function f^{-1} that we considered in Example 7.

EXERCISES 3.2

1–6 ■ The graph of a function f is shown. Determine whether f is one-to-one.

1.



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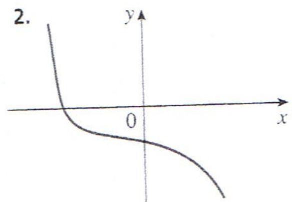

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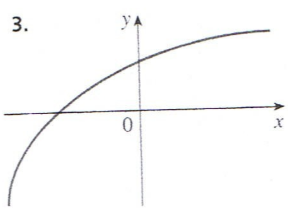
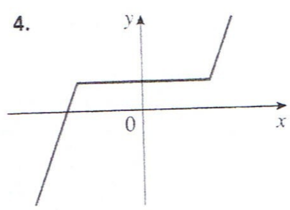


4.



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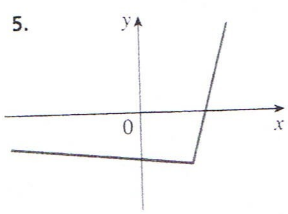
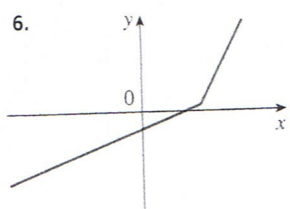


6.



7–12 ■ Determine whether the function is one-to-one.

7. $f(x) = 7x - 3$

8. $f(x) = x^2 - 2x + 5$

9. $g(x) = \sqrt{x}$

10. $g(x) = |x|$

11. $h(x) = x^4 + 5$

12. $h(x) = x^4 + 5, \quad 0 \leqslant x \leqslant 2$

13–18 ■ Show that f is one-to-one and find its inverse function.

13. $f(x) = 4x + 7$

14. $f(x) = \dfrac{x - 2}{x + 2}$

15. $f(x) = \dfrac{1 + 3x}{5 - 2x}$

16. $f(x) = 5 - 4x^3$

17. $f(x) = \sqrt{2 + 5x}$

18. $f(x) = x^2 + x, \quad x \geqslant -\frac{1}{2}$

19–24 ■

(a) Show that f is one-to-one.

(b) Use Theorem 7 to find $g'(a)$, where $g = f^{-1}$.

(c) Calculate $g(x)$ and state the domain and range of g.

(d) Calculate $g'(a)$ from the formula in part (c) and check that it agrees with the result of part (b).

(e) Sketch the graphs of f and g on the same axes.

19. $f(x) = 2x + 1, \quad a = 3$

20. $f(x) = 6 - x, \quad a = 2$

21. $f(x) = x^3, \quad a = 8$

22. $f(x) = \sqrt{x - 2}, \quad a = 2$

23. $f(x) = 9 - x^2, \quad 0 \leqslant x \leqslant 3, \quad a = 8$

24. $f(x) = 1/(x - 1), \quad x > 1, \quad a = 2$

25–30 ■ Find $g'(a)$, where g is the inverse function of the given function.

25. $f(x) = x^3 + x + 1, \quad a = 1$

26. $f(x) = x^5 - x^3 + 2x, \quad a = 2$

27. $f(x) = 3 + x^2 + \tan(\pi x/2), \quad -1 < x < 1, \quad a = 3$

28. $f(x) = \sqrt{x^3 + x^2 + x + 1}, \quad a = 2$

29. $f(x) = e^x, \quad a = 1$

30. $f(x) = 3 + x + e^x, \quad a = 4$

31. Suppose g is the inverse function of f and $f(4) = 5$, $f'(4) = \frac{2}{3}$. Find $g'(5)$.

32. Suppose g is the inverse function of a differentiable function f and let $G(x) = 1/g(x)$. If $f(3) = 2$ and $f'(3) = \frac{1}{9}$, find $G'(2)$.

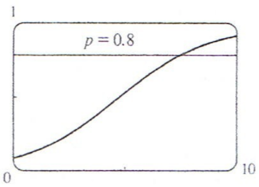
33–34 ■ Find an explicit formula for f^{-1} and use it to graph f^{-1}, f, and the line $y = x$ on the same screen. To check your work, see whether the graphs of f and f^{-1} are reflections in the line.

33. $f(x) = 1 - 2/x^2, \quad x > 0$

34. $f(x) = \sqrt{x^2 + 2x}, \quad x > 0$

49. $r = 1, -6$ **51.** $256e^{-2x}$ **53.** (b) -0.567143

55. (a) 1 (b) $kae^{-kt}/(1 + ae^{-kt})^2$
(c) $t \approx 7.4$ h



57. -1

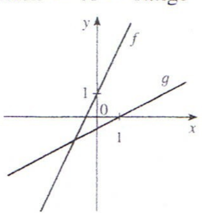
Exercises 3.2 ■ page 208

1. No **3.** Yes **5.** No **7.** Yes **9.** Yes **11.** No

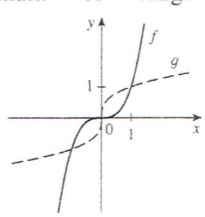
13. $f^{-1}(x) = (x - 7)/4$ **15.** $f^{-1}(x) = (5x - 1)/(2x + 3)$

17. $f^{-1}(x) = (x^2 - 2)/5, x \geq 0$

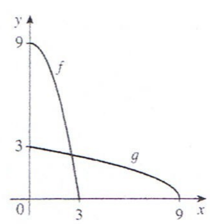
19. (b) $\frac{1}{2}$
(c) $g(x) = (x - 1)/2$,
domain $= \mathbb{R} =$ range
(e)



21. (b) $\frac{1}{12}$
(c) $g(x) = \sqrt[3]{x}$,
domain $= \mathbb{R} =$ range
(e)

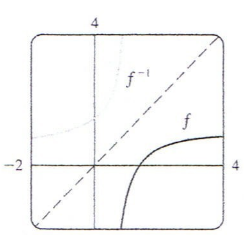


23. (b) $-\frac{1}{2}$
(c) $g(x) = \sqrt{9 - x}$, domain $= [0, 9]$,
range $= [0, 3]$
(e) See graph at right.

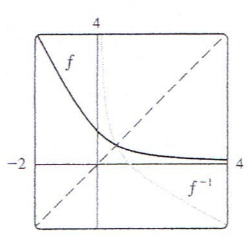


25. 1 **27.** $2/\pi$ **29.** 1 **31.** $\frac{3}{2}$

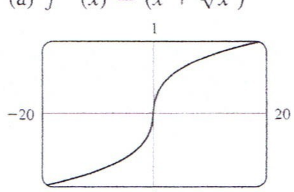
33. $f^{-1}(x) = \sqrt{2/(1 - x)}$



35.



37. (a) $f^{-1}(x) = (x + \sqrt[5]{x})^5$
(b)



Exercises 3.3 ■ page 214

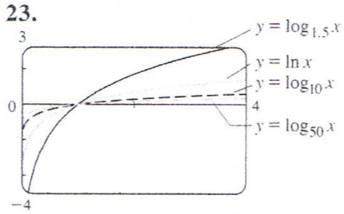
1. 6 **3.** $\frac{1}{3}$ **5.** -3 **7.** $\sqrt{2}$ **9.** 2 **11.** 1

13. 15 **15.** $\log_5(ab/c)$ **17.** $3 \ln 2$

19. $\ln[\sqrt[3]{x}/(2x + 3)^4]$

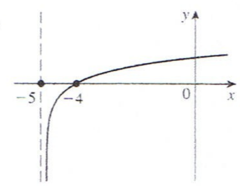
21. (a) 2.321928 (b) 2.025563 (c) 0.910239
(d) -7.399054

23.



All approach $-\infty$ as $x \to 0^+$, all pass through $(1, 0)$, and all increase. The larger the base, the slower the rate of increase for $x > 0$.

25.



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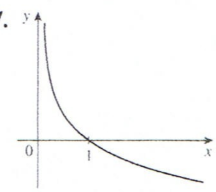

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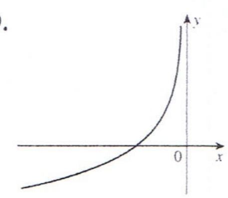

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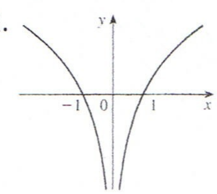

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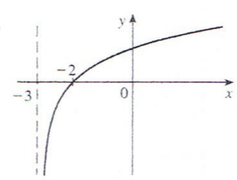

35. 8
37. $4 \ln 2$
39. $(e^3 + 1)/2$
41. $\log_3 m - 2$
43. 40
45. 3

47. e^e **49.** $\log_3(\log_2 5)$ **51.** 4 **53.** $(\ln C)/(a - b)$

55. 25.0855 **57.** -0.3319 **59.** About 1,084,588 mi

61. 8.3 **63.** $-\infty$ **65.** ∞ **67.** $-\infty$ **69.** 0

71. $(-\infty, 1), (-\infty, \infty)$ **73.** $(1, \infty), (-\infty, \infty)$

75. $y = e^x - 3$ **77.** $y = (\ln x)^2, x \geqslant 1$

79. $y = \log_{10}[x/(1 - x)]$

81. (b) $y = \frac{1}{2}(e^x - e^{-x})$ **83.** $\log_9 82$

87. $-1 \leq x < 1 - \sqrt{3}$ or $1 + \sqrt{3} < x \leq 3$

Exercises 3.4 ■ page 221

1. $f'(x) = 1/(x + 1), (-1, \infty), (-1, \infty)$

3. $f'(x) = 2x \ln(1 - x^2) - 2x^3/(1 - x^2), (-1, 1), (-1, 1)$

5. $f'(x) = 2x/[(x^2 - 4) \ln 3], |x| > 2, |x| > 2$

7. $y' = 1 + \ln x, y'' = 1/x$

9. $y' = 1/(x \ln 10), y'' = -1/(x^2 \ln 10)$

11. $f'(x) = (2 + \ln x)/(2\sqrt{x})$ **13.** $g'(x) = -2a/(a^2 - x^2)$