Trigonometric Functions

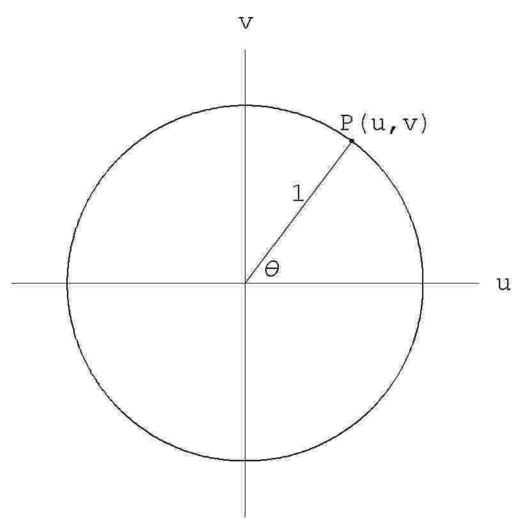
■ Essential facts

■ Radian Measure

■ The sine and cosine of any angle ϕ, $-\infty < \phi < \infty +$.

J11: pps 34-35. Modern Trigonometry: pps. 35-45

■ The unit circle



$\boldsymbol{u = \cos\theta}$

$\boldsymbol{v = \sin\theta}$

That is, by "$\cos\theta$" we mean the first component of the ordered pair (u, v) at P on the unit at the end of the terminal side of angle θ. Similarly for "$\sin\theta$".

Since any angle ϕ, $-\infty < \phi < \infty +$, can be mapped to some one angle θ, $0 \leq \theta \leq 2\pi$, the sine and cosine of every angle ϕ is defined.

■ The sine and cosine of a real number.

For a variety of reasons that will become clear as we study trigonometry and its applications, we desire to define the sine and cosine as functions whose domain is the set of all real numbers.

Figure 1 shows the unit circle at the origin of the coordinate system whose axes are u and v. We imagine that the real number line is tangent to the unit circle at the point $(u, v) = (1, 0)$ and that zero on the real number line coincides with $(u, v) = (1, 0)$.

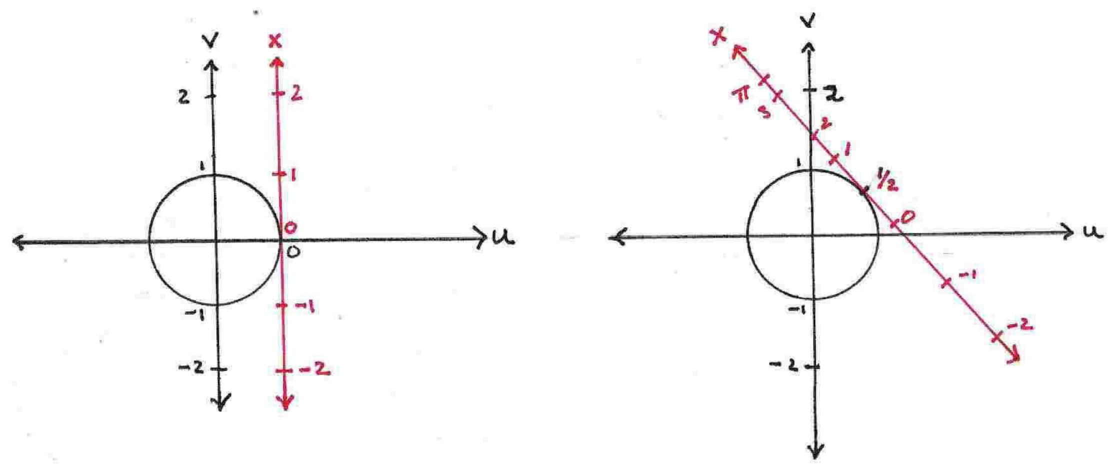


Figure 1

We next imagine that the real line is wrapped around the unit circle without slippage until the line is tangent to the circle at the desired point on the real number line. This wrapping is in a counterclockwise direction for a positive real number; it is in a clockwise direction for a negative real number. This wrapping is a function from the real numbers (domain) onto the points of the unit circle (range). Of course the function is not 1-1, because many real numbers map to the same point on the circle.

We can now define the sine and cosine as functions of a real number.

- ◆ Def. sine function = $\{(x, v) \ni x \in \mathbb{R}, (u, v)$ is the point on the unit circle corresponding to x.$\}$ We write $\sin x = v$ to indicate that the ordered pair (x, v) belongs to the sine function.

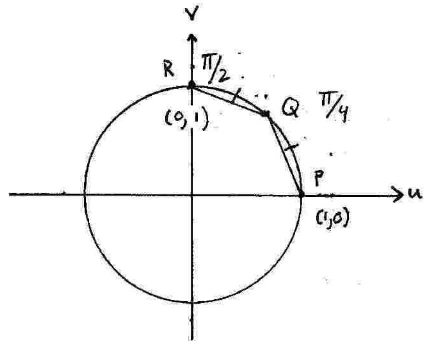
- ◆ Def. cosine function = $\{(x, u) \ni x \in \mathbb{R}, (u, v)$ is the point on the unit circle corresponding to x.$\}$ We write $\cos x = u$ to indicate that the ordered pair (x, u) belongs to the cosine function.

The definition of the sine function says that sine is a function that assigns to a real number the second coordinate of the point on the unit circle that corresponds to that real number.

The definition of the cosine function says that cosine is a function that assigns to a real number the first coordinate of the point on the unit circle that corresponds to that real number.

■ **Example**

Find $\cos \frac{\pi}{4}$ and $\sin \frac{\pi}{4}$.



Point Q is $Q(a, b)$. Note that $\cos \frac{\pi}{4} = a$ and $\sin \frac{\pi}{4} = b$. The lengths of arcs PQ and QR are each $\frac{\pi}{4}$. Since arcs of equal length subtend chords of equal length, line segments $\overline{PQ} = \overline{QR}$. Thus,

$$\overline{PQ} = \overline{QR}$$
$$\Longrightarrow \sqrt{(a-1)^2 + (b-0)^2 = (0-a)^2 + (1-b)^2}$$
$$\Longrightarrow a^2 - 2a + 1 + b^2 = a^2 + (1-b)^2$$
$$\Longrightarrow -2a = -2b$$
$$\Longrightarrow a = b$$

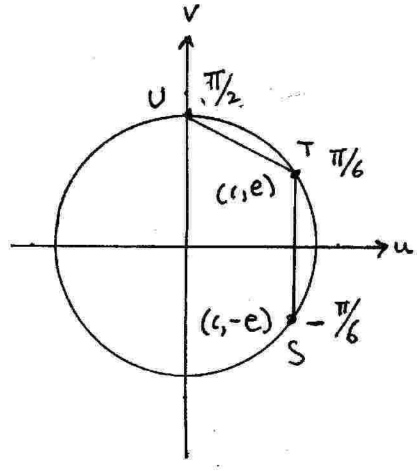
Then,

$a^2 + b^2 = 1$, since (a, b) is on the unit circle. Thus,

$a = \sin \frac{\pi}{4} = b = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$.

■ **Example**

Find $\sin \frac{\pi}{6}$ and $\cos \frac{\pi}{6}$.



If T has coordinates (c, e) then S has coordinates $(c, -e)$. $\overline{\text{TU}} = \overline{\text{ST}} = 2e$. Then,

$$\sqrt{(0-c)^2 + (1-e)^2} = 2e$$
$$\Longrightarrow c^2 + (1-e)^2 = 4e^2$$
$$\Longrightarrow 1 - 2e + 1 = 4e^2$$
$$\Longrightarrow 4e^2 + 2e - 2 = 0$$
$$\Longrightarrow (2e-1)(e+1) = 0$$
$$\Longrightarrow e = \frac{1}{2} \vee e = -1$$

Since point (c, e) is in the first quadrant, e must be positive. Therefore, $e = \frac{1}{2}$.
Then,

$c^2 + e^2 = 1$, since (c, e) is on the unit circle. Thus, $c^2 = 1 - e^2 = 1 - \frac{1}{4} = \frac{3}{4}$.

Therefore,

$c = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and $\sin \frac{\pi}{6} = \frac{1}{2}$.

Note: it is easier to recall the sine and cosine of $\frac{\pi}{4}$ by thinking of an isosceles right triangle, and it is easier to recall the sines and cosines of $\frac{\pi}{3}$ and $\frac{\pi}{6}$ by thinking of a 30-60-90 triangle.