Math 11 - Answers to J11-BA page 85 Exercises
a.k.a *Take Home Exam*

These have been checked

J11 Exercises p 85

[1]

Solution A. Since $S_3 = S_5$, $a_1 + a_2 + a_3 = a_1 + a_2 + a_3 + a_4 + a_5$. But this means $(a_4 + a_5)$ must contribute nothing to the sum. Thus, $a_4 + a_5 = 0 \Longrightarrow (5 + 3d) + (5 + 4d) = 0 \Longrightarrow 7d = -10 \Longrightarrow d = \frac{-10}{7}$.

Solution B. Use $S_n = \frac{n}{2}[2a + (n-1)d]$. Thus, $S_3 = \frac{3}{2}[10 + 2d] = \frac{5}{2}[10 + 4d] = S_5 \Longrightarrow d = \frac{-10}{7}$.

[2.1]

$$a_5 = a_1 + 4d = 108$$
$$a_{20} = a_1 + 19d = -237$$

Then, $a_1 = 200$, $d = -23$

[2.2]

$$S_n = \frac{n[2 \cdot 200 - 23(n-1)]}{2} = \frac{1}{2}(423n - 23n^2) = \frac{n}{2}(423 - 23n)$$

The roots of $y = \frac{x}{2}(423 - 23x)$are $x = 0$ and $x = \frac{423}{23}$. The extreme value, in this case its maximum, occurs at $x = \frac{1}{2}\left(\frac{423}{23}\right) = 9\frac{9}{46}$. Therefore, $n = 9$. [Note: since $9\frac{9}{46}$ is closer to 9 than to 10, so the maxi mum of S_n is at 9, not 10.]

[3]

$$a_3 = ar^2 = 12$$
$$a_6 = ar^5 = 96$$

Solving simultaneously, $\frac{12}{r^2}(r^5) = 96 \Longrightarrow r^3 = 8 \Longrightarrow r = 2$. $a = \frac{12}{4} = 3$. So, $a_n = 3 \cdot 2^{n-1}$. Then,

[3.1]

$\sum_{k=1}^{n} (3 \cdot 2^{k-1})^2 = \sum_{k=1}^{n} 9 \cdot 2^{2(k-1)} = \sum_{k=1}^{n} 9 \cdot 4^{(k-1)}$.

This is the sum of a geometric series with $a = 9$ and $r = 4$. Thus, the sum of the first n terms is

$S_n = 9\left(\frac{4^n - 1}{4 - 1}\right) = 3(4^n - 1)$

[3.2]

Product of first *n* terms:

$$\prod_{i=1}^{n} (3 \times 2^{i-1})$$

$$= [(3 \times 2^0)(3 \times 2^1)(3 \times 2^2)(3 \times 2^3) \times \cdots \times (3 \times 2^{n-1})]$$

$$= 3^n (2^1 \times 2^2 \times 2^3 \times \cdots \times 2^{n-1})$$

$$= 3^n 2^{[1+2+3+\cdots+(n-1)]}$$

$$= 3^n 2^{\frac{n(n-1)}{2}}$$

[4.1]

$$\sum_{i=1}^{n} (1 + 3(n-1))^2 = \sum_{i=1}^{n} (4 - 12i + 9i^2) = \sum_{i=1}^{n} 4 - \sum_{i=1}^{n} 12i + \sum_{i=1}^{n} 9i^2$$

$$= 4n - 12\left(\frac{n(n+1)}{2}\right) + 9\left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$= \frac{n}{2}(6n^2 - 3n - 1)$$

[4.2]

$$\sum_{i=1}^{n} (2i)(3 + (i-1))^2 = \sum_{i=1}^{n} (8i + 8i^2 + 2i^3) = 2\sum_{i=1}^{n} i^3 + 8\sum_{i=1}^{n} i^2 + 8\sum_{i=1}^{n} i$$

$$= 2\left(\frac{n(n+1)}{2}\right)^2 + 8\left(\frac{n(n+1)(2n+1)}{6}\right) + 8\left(\frac{n(n+1)}{2}\right)$$

$$= \frac{n}{6}(32 + 51n + 22n^2 + 3n^3)$$

[5]

$\{a_n\} = 2, 3, 9, 18, 28, 37, 43, 44, \ldots$

$\{b_n\} = 1, 6, 9, 10, 9, 6, 1, \ldots$

$\{c_n\} = 5, 3, 1, -1, -3, -5, \ldots$

$$c_n = 5 - 2(n-1)$$

$$b_n = 1 + \sum_{k=1}^{n-1} (5 - 2(k-1))$$

$$= 1 + \sum_{k=1}^{n-1} (7 - 2k)$$

$$= (1 + 7(n-1) - n(n-1))$$

$$\therefore b_n = -6 + 8n - n^2$$

$$a_n = 2 + \sum_{k=1}^{n-1} (-6 + 8k - k^2)$$

$$\therefore a_n = \frac{1}{6}(48 - 61n + 27n^2 - 2n^3)$$

[6]

Let S_n be the sum of the first n terms of $\{a_n\}$. If $S_n = an^2 + bn + c$, then is $\{a_n\}$ an arithmetic sequence?

Suppose $S_n = an^2 + bn + c$. Then $S_{n-1} = a(n-1)^2 + b(n-1) + c$. Thus,

$$a_n = s_n - s_{n-1} = an^2 + bn + c - (a(n-1)^2 + b(n-1) + c)$$

$$a_n = (b - a) + (2a)n$$

$\therefore$ This is an arithmetic sequence with first term $(b - a)$ and common difference $2a$. That is,

$\{(b-a) + (n-1)(2a)\}$