[P1-9-05-09-23]

Example 2. $\sqrt{517} \cdot \sqrt{517} = 517.$

Example 3. $\sqrt{7} \cdot \sqrt{6} = \sqrt{42}.$

Example 4. $\sqrt{7} \cdot \sqrt{14} = \sqrt{7} \cdot \sqrt{7} \cdot \sqrt{2}$
$= 7\sqrt{2}$

or $\sqrt{7} \cdot \sqrt{14} = \sqrt{7 \cdot 7 \cdot 2}$
$= 7\sqrt{2}.$

Example 5. $3\sqrt{6} \cdot 5\sqrt{96}$
$= 15\sqrt{6} \cdot \sqrt{16} \cdot \sqrt{6}$
$= 15 \cdot 6 \cdot 4$
$= 360$

or $3\sqrt{6} \cdot 5\sqrt{96}$
$= 15\sqrt{6 \cdot 16 \cdot 6}$
$= 15 \cdot 6 \cdot 4$
$= 360.$

Example 6. $\sqrt{\frac{5}{6}} \cdot \sqrt{\frac{25}{6}} = \sqrt{\frac{25 \cdot 5}{6 \cdot 6}}$
$= \frac{5\sqrt{5}}{6},$ or $\frac{5}{6}\sqrt{5}.$

Example 7. $5\sqrt{6} \cdot 7\sqrt{50}$
$= 35 \cdot \sqrt{2} \cdot \sqrt{3} \cdot \sqrt{25} \cdot \sqrt{2}$
$= 35 \cdot 2 \cdot 5 \cdot \sqrt{3}$
$= 350\sqrt{3}$

or $5\sqrt{6} \cdot 7\sqrt{50}$
$= 35\sqrt{2 \cdot 3 \cdot 25 \cdot 2}$
$= 35 \cdot 2 \cdot 5 \cdot \sqrt{3}$
$= 350\sqrt{3}.$

Example 8. $\sqrt{\frac{15}{7}} \cdot \sqrt{\frac{21}{5}}$
$= \sqrt{\frac{3 \cdot 5 \cdot 3 \cdot 7}{7 \cdot 5}}$
$= 3.$

Example 9. $\sqrt{a^3} \cdot \sqrt{a^4}$
$= \sqrt{a^3 \cdot a^3 \cdot a}$
$= a^3\sqrt{a}.$

Exercises [A]

Multiply. The radicand in the product should be as small as possible.

1. $\sqrt{6} \cdot \sqrt{6}$
2. $\sqrt{7} \cdot \sqrt{7}$
3. $\sqrt{6} \cdot \sqrt{2}$
4. $\sqrt{8} \cdot \sqrt{3}$
5. $\sqrt{7} \cdot \sqrt{14}$
6. $\sqrt{5} \cdot \sqrt{6}$
7. $\sqrt{3} \cdot \sqrt{3}$
8. $\sqrt{10} \cdot \sqrt{4}$
9. $\sqrt{3} \cdot \sqrt{27}$
10. $\sqrt{15} \cdot \sqrt{5}$
11. $\sqrt{2} \cdot \sqrt{18}$
12. $\sqrt{6} \cdot \sqrt{\frac{2}{3}}$
13. $\sqrt{\frac{2}{3}} \cdot \sqrt{\frac{3}{2}}$
14. $\sqrt{\frac{5}{7}} \cdot \sqrt{\frac{14}{5}}$
15. $\sqrt{18} \cdot \sqrt{32}$
16. $\sqrt{17} \cdot \sqrt{51}$
17. $\sqrt{5} \cdot \sqrt{\frac{3}{5}}$
18. $22 \cdot 2\sqrt{5}$
19. $30 \cdot 5\sqrt{7}$
20. $19 \cdot 2\sqrt{11}$
21. $\sqrt{a} \cdot \sqrt{a^2}$
22. $\sqrt{a^3} \cdot \sqrt{a^4}$
23. $\sqrt{2\,ab} \cdot \sqrt{3\,ab}$
24. $\sqrt{5\,a^2c} \cdot \sqrt{ac}$
25. $\sqrt{x^3} \cdot \sqrt{x^3}$
26. $\sqrt{b} \cdot \sqrt{2\,b}$
27. $\sqrt{45} \cdot \sqrt{80}$
28. $\sqrt{27} \cdot \sqrt{108}$
29. $\sqrt{10} \cdot \sqrt{125}$
30. $\sqrt{8} \cdot \sqrt{6} \cdot \sqrt{3}$
31. $\sqrt{14} \cdot \sqrt{7} \cdot \sqrt{21}$
32. $\sqrt{x} \cdot \sqrt{\frac{4\,a}{x}}$

33. $\sqrt{98\,x} \cdot \sqrt{\frac{1}{2}\,x}$
34. $\sqrt{\frac{10}{11}} \cdot \sqrt{\frac{22}{5}}$
35. $\sqrt{\frac{14}{27}} \cdot \sqrt{\frac{4}{21}}$
36. $\sqrt{\frac{96}{25}} \cdot \sqrt{\frac{125}{54}}$
37. $\sqrt{3\,b} \cdot \sqrt{6\,b} \cdot \sqrt{b}$
38. $4\sqrt{10} \cdot 5\sqrt{12}$
39. $3\sqrt{5} \cdot 2\sqrt{5}$
40. $7\sqrt{21} \cdot 3\sqrt{3}$
41. $20\sqrt{20} \cdot 5\sqrt{5}$
42. $6\sqrt{14} \cdot 2\sqrt{7}$
43. $8\,a\sqrt{5\,a} \cdot 3\sqrt{10\,a}$
44. $3\sqrt{6} \cdot 5\sqrt{5} \cdot 2\sqrt{15}$
45. $(5\sqrt{3})^2$
46. $(2\sqrt{6})^2$
47. $(2\sqrt{12})^2$
48. $(5\sqrt{3\,x})^2$
49. $5(\sqrt{3\,x})^2$
50. $6\sqrt{(3\,x)^2}$
51. $\sqrt{5(3\,x)} \cdot 5\sqrt{3\,x}$
52. $7(3\sqrt{7})^2$
53. $5\,x^2\sqrt{5\,x^2} \cdot 5\sqrt{x}$

The direction "simplify" in the following exercises means "change the form of the radical so that the radicand of the new form will be an integer and as small as possible." Study the following examples before doing the exercises.

Example 1. Simplify $\sqrt{72\,a^5b^5}$.

Solution:
$$\begin{aligned}\sqrt{72\,a^5b^5} &= \sqrt{36\,a^4b^4 \cdot 2\,ab}\\ &= \sqrt{36\,a^4b^4} \cdot \sqrt{2\,ab}\\ &= 6\,ab^2\sqrt{2\,ab}\end{aligned}$$

Example 2. Simplify $15\sqrt{27}$.

Solution:
$$\begin{aligned}15\sqrt{27} &= 15 \cdot \sqrt{9} \cdot \sqrt{3}\\ &= 15 \cdot 3 \cdot \sqrt{3}\\ &= 45\sqrt{3}\end{aligned}$$

Example 3. Simplify $4 \cdot \sqrt{\frac{75}{64}}$.

Solution:
$$\begin{aligned}4 \cdot \sqrt{\tfrac{75}{64}} &= 4 \cdot \sqrt{\tfrac{25}{64}} \cdot \sqrt{3}\\ &= 4 \cdot \tfrac{5}{8} \cdot \sqrt{3}\\ &= \frac{5\sqrt{3}}{2}\end{aligned}$$

In Example 1 above and in some of the following exercises we are asked to simplify expressions of the form $\sqrt{a^2}$. It is readily seen that if $a \geqq 0$, then $\sqrt{a^2} = a$. However, if $a = -2$, we have $\sqrt{a^2} = \sqrt{(-2)^2} = \sqrt{4} = 2$. Here $a = -2$ and $\sqrt{a^2} = 2$, so that $\sqrt{a^2} = -a$. This conclusion holds for all negative values of a. Thus,

if $a \geqq 0$, we have $\sqrt{a^2} = a$;
and if $a < 0$, we have $\sqrt{a^2} = -a$.

To remove the need for writing two forms of answer in simplifying $\sqrt{a^2}$, we shall consider that any letter under a radical sign is representing a positive number. With this restriction, we may write a as the simplified form of $\sqrt{a^2}$.

Exercises [A–1]

Simplify:

1. $\sqrt{12}$
2. $\sqrt{32}$
3. $\sqrt{24}$
4. $\sqrt{36}$
5. $\sqrt{48}$
6. $\sqrt{54}$
7. $\sqrt{68}$
8. $\sqrt{76}$
9. $\sqrt{84}$
10. $\sqrt{250}$
11. $\sqrt{16\,a}$
12. $\sqrt{98}$
13. $15\sqrt{18}$
14. $12\sqrt{32}$
15. $2\sqrt{200}$
16. $6\sqrt{50}$
17. $\sqrt{\frac{44}{2}}$
18. $\sqrt{196}$
19. $\sqrt{50\,m^3n}$
20. $\sqrt{450}$
21. $\frac{2}{5}\sqrt{45}$
22. $\sqrt{8\,a^2}$
23. $\sqrt{90\,m^5}$
24. $\sqrt{120\,a^2b^3}$
25. $\sqrt{\frac{16}{25}}$
26. $\sqrt{\frac{7}{9}}$
27. $\frac{1}{2}\cdot\sqrt{\frac{5}{16}}$
28. $5\cdot\sqrt{\frac{24}{25}}$
29. $14\cdot\sqrt{\frac{8}{49}}$

Exercises [A–2]

Simplify:

1. $\sqrt{8}$
2. $\sqrt{20}$
3. $\sqrt{28}$
4. $\sqrt{44}$
5. $\sqrt{52}$
6. $\sqrt{64\,r^4}$
7. $\sqrt{72}$
8. $\sqrt{80}$
9. $\sqrt{125}$
10. $\sqrt{75\,x}$
11. $\sqrt{18\,b^2}$
12. $5\sqrt{8}$
13. $6\sqrt{25}$
14. $8\sqrt{27}$
15. $4\sqrt{4}$
16. $10\sqrt{99}$
17. $\frac{1}{6}\sqrt{300}$
18. $\sqrt{40\,c^3}$
19. $\sqrt{60\,a^4b}$
20. $\sqrt{128}$
21. $\sqrt{27\,x^3}$
22. $\sqrt{88\,ab^3}$
23. $\sqrt{289\,a^2}$
24. $\sqrt{x^3}$
25. $\sqrt{\frac{3}{4}}$
26. $\sqrt{\frac{4}{81}}$
27. $\sqrt{\frac{49}{9}}$

Exercises [B]

Simplify:

1. $\sqrt{724}$
2. $\sqrt{529}$
3. $\sqrt{816}$
4. $\sqrt{162}$
5. $\sqrt{\frac{40}{4}}$
6. $\sqrt{4\,y^3}$
7. $\sqrt{\frac{20}{9}}$
8. $\sqrt{\frac{32}{49}}$
9. $\sqrt{\frac{200}{16}}$
10. $\sqrt{\frac{98}{36}}$
11. $8\sqrt{\frac{5}{64}}$
12. $\sqrt{75\,b^4}$
13. $\sqrt{2\,x^2+4\,x+2}$
14. $\sqrt{3(a-1)^2}$
15. $\sqrt{5\,a^2-30\,ab+45\,b^2}$
16. $\sqrt{(p-q)(p+q)^2}$

Exercises [A–1]

Carry out the indicated operations. Simplify results as far as possible.

1. $\sqrt{2}(\sqrt{2}+1)$
2. $2\sqrt{3}(2-\sqrt{3})$
3. $\sqrt{5}(1+\sqrt{5})$
4. $\sqrt{2}(2\sqrt{5}-\sqrt{8})$
5. $\sqrt{6}(2\sqrt{2}-\sqrt{3})$
6. $(\sqrt{15}+\sqrt{5})/\sqrt{5}$
7. $(2\sqrt{3}-\sqrt{6})/\sqrt{3}$
8. $\sqrt{\frac{128}{3}}\cdot\sqrt{\frac{15}{8}}$
9. $(2\sqrt{3}+3\sqrt{2})(3\sqrt{3}-\sqrt{2})$
10. $(\sqrt{6}-\sqrt{3})^2$
11. $(\sqrt{3}+1)(\sqrt{3}-1)$
12. $(\sqrt{3}+\sqrt{2})^2$
13. $(3-\sqrt{2})^2$
14. $(3+\sqrt{2})(3-\sqrt{2})$
15. $(\sqrt{14}-\sqrt{7})^2$
16. $[(\sqrt{3}-1)/3]^2$
17. $\frac{\sqrt{5}+2}{3}\cdot\frac{\sqrt{5}-2}{3}$
18. $\left(\frac{\sqrt{6}+1}{2}\right)^2$
19. $(\sqrt{40}+\sqrt{10})\div\sqrt{10}$
20. $\frac{-2+\sqrt{20}}{-2}$
21. $\frac{-5+2\sqrt{50}}{15}$

Exercises [A–2]

Carry out the indicated operations. Simplify results as far as possible.

1. $\sqrt{3}(2\sqrt{3}+4)$
2. $\sqrt{2}(6+\sqrt{2})$
3. $3\sqrt{3}(\sqrt{2}+\sqrt{3})$
4. $\sqrt{5}(\sqrt{5}+\sqrt{10})$
5. $2\sqrt{3}(4-\sqrt{21})$
6. $(\sqrt{2}+2\sqrt{6})\div\sqrt{2}$
7. $\sqrt{\frac{5}{14}}\cdot\sqrt{\frac{7}{10}}$
8. $\sqrt{\frac{18}{13}}\div\sqrt{\frac{54}{65}}$
9. $(\sqrt{5}-1)(\sqrt{5}+1)$
10. $(\sqrt{5}-2)^2$
11. $(\sqrt{2}+2)(\sqrt{2}-2)$
12. $(2+\sqrt{3})^2$
13. $(\sqrt{6}+\sqrt{2})^2$
14. $(\sqrt{6}+\sqrt{3})(\sqrt{6}-\sqrt{3})$
15. $\frac{\sqrt{3}-1}{2}\cdot\frac{\sqrt{3}+1}{2}$
16. $\left(\frac{1+\sqrt{2}}{2}\right)^2$
17. $\left(\frac{2-\sqrt{5}}{3}\right)^2$
18. $(\sqrt{15}-\sqrt{19})/\sqrt{5}$
19. $\frac{-10+2\sqrt{10}}{2}$
20. $\frac{6+\sqrt{18}}{6}$
21. $\frac{16+32\sqrt{28}}{4}$

Exercises [B]

Carry out the indicated operations. Simplify results as far as possible.

1. $(2\sqrt{5}+\sqrt{15})(\sqrt{5}-\sqrt{15})$
2. $(3\sqrt{2}-\sqrt{10})(2\sqrt{2}+\sqrt{10})$
3. $\sqrt{(5-\sqrt{2})(5+\sqrt{2})}$
4. $\sqrt{(\sqrt{6}-\sqrt{2})(\sqrt{6}+\sqrt{2})}$
5. $(5\sqrt{7}-1)(2\sqrt{7}+3)$
6. $\dfrac{2\sqrt{48}-\sqrt{96}}{4}$
7. $\dfrac{\sqrt{45}+\sqrt{20}-2\sqrt{125}}{6}$
8. $\dfrac{-11-22\sqrt{99}}{11}$

Rational Approximations to Irrational Numbers

The ability to add, subtract, multiply, and divide expressions involving radicals is necessary in mathematically exact work. When a radical occurs in a situation involving measurement, however, an engineer or mechanic commonly uses a decimal value which is sufficiently close to the radical for his purpose.

Suppose, for example, that a mechanic needs a square sheet of metal with area 5 square feet. He would probably find from a chart that the side of the square should be 2 feet $2\frac{7}{8}$ inches. This corresponds closely to 2.24 feet and the square of 2.24 is just a little over 5. So 2.24 is close to $\sqrt{5}$ and it gives the mechanic a good working value for the side of his square.

The following sets of numbers are decimal values which give increasingly close approximations to $\sqrt{5}$.

$$\begin{aligned} 2.3 &> \sqrt{5} > 2.2 \\ 2.24 &> \sqrt{5} > 2.23 \\ 2.237 &> \sqrt{5} > 2.236 \\ 2.2361 &> \sqrt{5} > 2.2360. \end{aligned}$$

The inequalities may be verified by squaring the numbers that are expressed as decimals.

The table on page 439 lists rational numbers which are approximations to $\sqrt{N}$ when N is a member of the set of integers from 1 through 100. Each approximation is taken to three places of decimals. For example, $\sqrt{99}$ is given as 9.950. This means that the square of 9.950 is closer to 99 than is the square of either 9.951 or 9.949.

The fact that each number given in the table is closer to the corresponding $\sqrt{N}$ than any other number of three decimal places is expressed by saying that the numbers in the table are "rounded off" to the nearest thousandth.

For Exs. 2-21, R = rational, and I = irrational.

2. R
3. R
4. R
5. I
6. R
7. R
8. I
9. I
10. R
11. R
12. R
13. R
14. R
15. I
16. I
17. R
18. R
19. R
20. R
21. R
22. $\frac{7}{2}$
23. $\frac{3}{8}$
24. $\frac{23}{5}$
25. $\frac{25}{4}$

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3. 1
4. $\sqrt{5}$
5. 12
6. 17
7. $\sqrt{39}$
8. 4
9. $\sqrt{10}$

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14. $\sqrt{2}$
15. 24
16. $17\sqrt{3}$
17. $\sqrt{3}$
18. $44\sqrt{5}$
19. $150\sqrt{7}$
20. $38\sqrt{11}$
21. $a\sqrt{a}$
22. $a^3\sqrt{a}$
23. $ab\sqrt{6}$
24. $ac\sqrt{5a}$
25. x^3
26. $b\sqrt{2}$
27. 60
28. 54
29. $25\sqrt{2}$
30. 12
31. $7\sqrt{42}$
32. $2\sqrt{a}$
33. $7x$
34. 2
35. $\frac{2\sqrt{2}}{9}$
36. $\frac{4\sqrt{5}}{3}$
37. $3b\sqrt{2b}$
38. $40\sqrt{30}$
39. 30
40. $63\sqrt{7}$
41. 1000
42. $84\sqrt{2}$
43. $120a^2\sqrt{2}$
44. $450\sqrt{2}$
45. 75
46. 24
47. 48
48. $75x$
49. $15x$
50. $18x$
51. $15x\sqrt{5}$
52. 441
53. $25x^3\sqrt{5x}$

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9. $2\sqrt{21}$
10. $5\sqrt{10}$
11. $4\sqrt{a}$
12. $7\sqrt{2}$
13. $45\sqrt{2}$
14. $48\sqrt{2}$
15. $20\sqrt{2}$
16. $30\sqrt{2}$
17. $\sqrt{22}$
18. 14
19. $5m\sqrt{2mn}$
20. $15\sqrt{2}$
21. $\frac{6}{5}\sqrt{5}$
22. $2a\sqrt{2}$
23. $3m^2\sqrt{10m}$
24. $2ab\sqrt{30b}$
25. $\frac{4}{5}$
26. $\frac{1}{3}\sqrt{7}$
27. $\frac{1}{8}\sqrt{5}$
28. $2\sqrt{6}$
29. $4\sqrt{2}$

Page 432 (middle)

1. $2\sqrt{2}$
2. $2\sqrt{5}$
3. $2\sqrt{7}$
4. $2\sqrt{11}$
5. $2\sqrt{13}$
6. $8r^2$
7. $6\sqrt{2}$
8. $4\sqrt{5}$
9. $5\sqrt{5}$
10. $5\sqrt{3x}$
11. $3b\sqrt{2}$
12. $10\sqrt{2}$
13. 30
14. $24\sqrt{3}$
15. 8
16. $30\sqrt{11}$
17. $\frac{5}{3}\sqrt{3}$
18. $2c\sqrt{10c}$
19. $2a^2\sqrt{15b}$
20. $8\sqrt{2}$
21. $3x\sqrt{3x}$
22. $2b\sqrt{22ab}$
23. $17a$
24. $x\sqrt{x}$
25. $\frac{1}{2}\sqrt{3}$
26. $\frac{2}{9}$
27. $\frac{7}{3}$

Page 432 (bottom)

1. $2\sqrt{181}$
2. 23
3. $4\sqrt{51}$
4. $9\sqrt{2}$
5. $\sqrt{10}$
6. $2y\sqrt{y}$
7. $\frac{2}{3}\sqrt{5}$
8. $\frac{4}{7}\sqrt{2}$
9. $\frac{5}{2}\sqrt{2}$
10. $\frac{7}{6}\sqrt{2}$
11. $\sqrt{5}$
12. $5b^2\sqrt{3}$
13. $(x + 1)\sqrt{2}$
14. $(a - 1)\sqrt{3}$
15. $(a - 3b)\sqrt{5}$
16. $(p + q)\sqrt{p - q}$

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1. $4\sqrt{2}$
2. $-4\sqrt{5}$
3. $6\sqrt{6}$
4. $4\sqrt{\frac{3}{2}}$
5. $-\sqrt{6}$
6. $\frac{5}{2}\sqrt{2}$
7. $\sqrt{3} + 2\sqrt{2}$
8. $2\sqrt{3} + 3\sqrt{2}$
9. $-\sqrt{5}$
10. 0
11. $7\sqrt{2}$
12. $4\sqrt{7}$
13. $17\sqrt{2}$
14. $\sqrt{5}$
15. $-\sqrt{10}$
16. $14\sqrt{3}$
17. $3\sqrt{3} - 2\sqrt{30}$
18. $3a\sqrt{2}$
19. $4x$
20. $3\sqrt{b}$
21. $\sqrt{c} + \sqrt{2c}$
22. $2y\sqrt{3}$
23. $x\sqrt{2} - x$
24. $2\sqrt{6}$
25. $4\sqrt{2}$
26. $2\sqrt{5} + 5\sqrt{\frac{10}{3}}$
27. $\frac{19}{12}\sqrt{5}$
28. $\frac{62}{3}\sqrt{3}$
29. $5\sqrt{14} - 4\sqrt{7}$
30. 1

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1. 2
2. $\sqrt{3}$
3. $\sqrt{2}$
4. $\sqrt{5}$
5. $2\sqrt{7}$
6. $\sqrt{6}$
7. $\frac{3}{2}$
8. 2
9. $\sqrt{3a}$
10. $\sqrt{b}$
11. $a^2\sqrt{2}$
12. $2\sqrt{5}$
13. $\sqrt{3}$
14. $\sqrt{7}$
15. $\sqrt{3}$
16. $\sqrt{3}$
17. $y\sqrt{5}$
18. $\frac{2}{3}$
19. 2
20. 3
21. $\frac{1}{6}\sqrt{7}$

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1. $2 + \sqrt{2}$
2. $4\sqrt{3} - 6$
3. $\sqrt{5} + 5$
4. $2\sqrt{10} - 4$
5. $4\sqrt{3} - 3\sqrt{2}$
6. $\sqrt{3} + 1$
7. $2 - \sqrt{2}$
8. $4\sqrt{5}$
9. $12 + 7\sqrt{6}$
10. $9 - 6\sqrt{3}$
11. 2
12. $5 + 2\sqrt{6}$