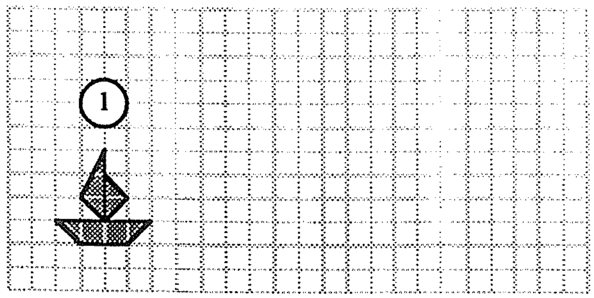
CJ8 04-01-12

CHAP 5

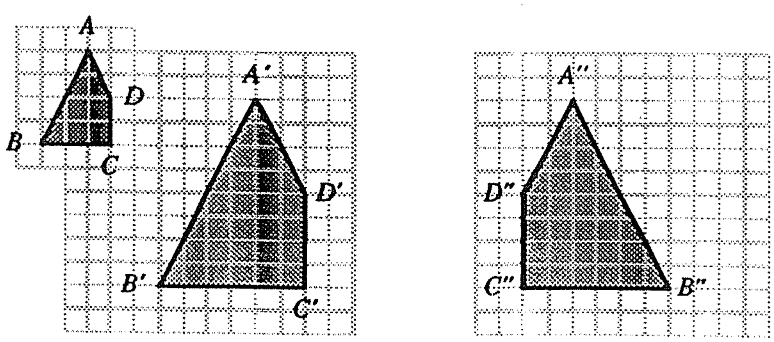
1 SIMILAR FIGURES

1 Similar Figures

Problem 1 Draw an enlargement of figure 1 on the graph paper section to the right, doubling the dimensions of the original figure.



When we enlarge or reduce a figure by a fixed proportion without changing its shape, the new figure and the original figure are said to be **similar**.

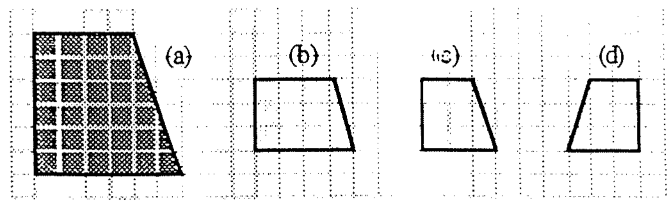


Quadrilaterals *ABCD* and *A'B'C'D'* above are similar. When two figures are similar, features such as vertices *A* and *A'*, sides *AB* and *A'B'*, and $\angle A$ and $\angle A'$ are said to correspond to each other.

Quadrilateral $A''B''C''D''$ is a mirror image of quadrilateral $A'B'C'D'$. Here, too, quadrilaterals $ABCD$ and $A''B''C''D''$ are similar. A and A'', B and B'', C and C'', and D and D'' are corresponding vertices.

Problem 2 Given quadrilaterals $ABCD$ and $A''B''C''D''$ above, identify the corresponding sides and angles for AB, BC, CD, DA, and $\angle A$, $\angle B$, $\angle C$, $\angle D$.

Problem 3 Which of the quadrilaterals in (b)–(d) below are similar to quadrilateral (a)?



We can express the fact that two quadrilaterals $ABCD$ and $A'B'C'D'$ are similar by

quadrilateral $ABCD$ ~ quadrilateral $A'B'C'D'$

The symbol ~ stands for "is similar to." When we use this symbol with polygons, we give the corresponding vertices in the same order for each figure.

If we compare the lengths of the corresponding sides in quadrilaterals $ABCD$ and $A'B'C'D'$, we find that

$$AB : A'B' = BC : B'C' = CD : C'D' = DA : D'A' = 1 : 2$$

Thus,

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{DA}{D'A'} = \frac{1}{2} \qquad (1)$$

Comparing the corresponding angles,

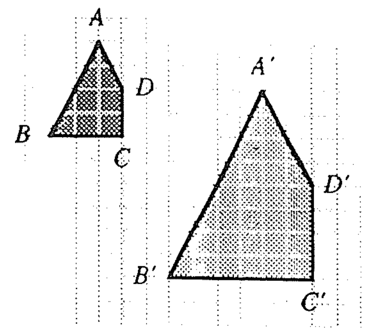
$$\angle A = \angle A', \ \angle B = \angle B', \ \angle C = \angle C', \ \angle D = \angle D' \qquad (2)$$

These facts allow us to formulate the following general properties of similar figures.

Properties of Similar Figures

In similar figures, the ratios of the lengths of corresponding sides are equal, and the measures of corresponding angles are equal.

The ratio of the lengths of corresponding sides in similar figures, or the value of the ratio, is called the **similarity ratio**. For example, in the diagram to the right the similarity ratio of quadrilaterals $ABCD$ and $A'B'C'D'$ is said to be $1 : 2$, or $\frac{1}{2}$.

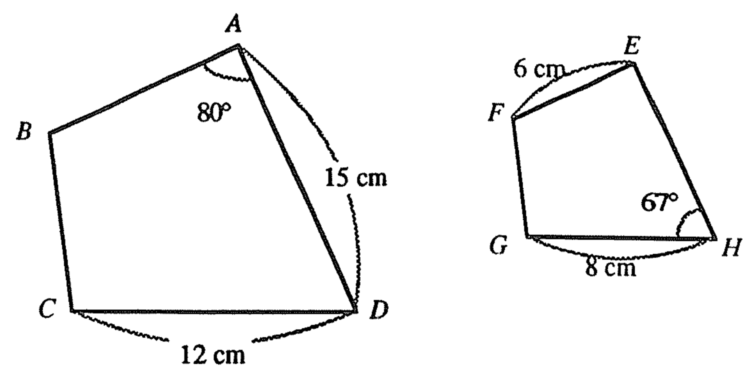


Any two circles are similar, and their similarity ratio is equal to the ratio of their radii.

Congruent figures can be regarded as similar figures with a similarity ratio of 1 : 1.

Problem 4 Given that $\triangle ABC \sim \triangle DEF$, identify the corresponding vertices, sides, and angles. Assuming that the similarity ratio for $\triangle ABC$ and $\triangle DEF$ is 1 : 3, write expressions like (1) and (2) above.

Problem 5 In the diagram below, assume that quadrilateral $ABCD \sim$ quadrilateral $EFGH$.



(1) Give the similarity ratio for quadrilaterals $ABCD$ and $EFGH$.

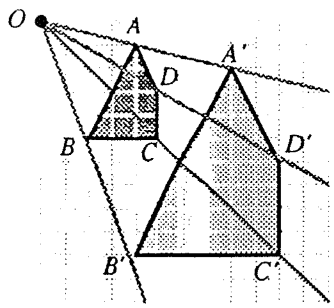
(2) Find the lengths of sides AB and EH.

(3) Find the measures of $\angle D$ and $\angle E$.

Position of Similarity and Center of Similarity

If we place the similar quadrilaterals $ABCD$ and $A'B'C'D'$ from page 157 as in the diagram below, lines AA', BB', CC', and DD' intersect at a single point O, and if we extend OA, OB, OC, and OD to twice their length, then we obtain OA', OB', OC', and OD'.

We can think of quadrilateral $A'B'C'D'$ as a figure drawn by connecting in succession the points A', B', C', and D' arrived at in this way.

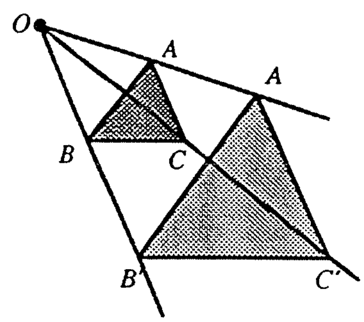


Now let's draw $\Delta A'B'C'$ similar to ΔABC in the same way. In order to ensure that the similarity ratio for ΔABC and $\Delta A'B'C'$ will be 1 : 2, we can carry out the following steps.

Plot points A', B', and C' on rays from point O through the vertices of ΔABC such that

$$\frac{OA}{OA'} = \frac{OB}{OB'} = \frac{OC}{OC'} = \frac{1}{2}$$

and connect these points successively with line segments.

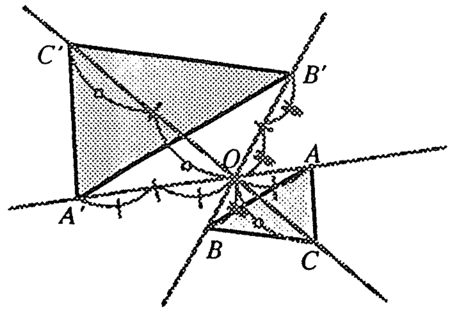


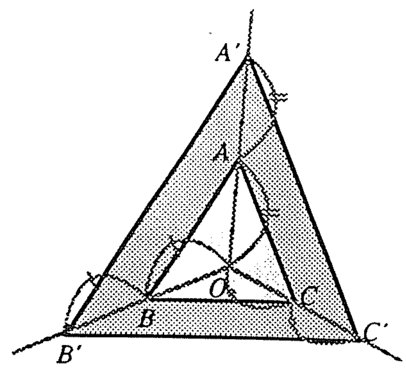
Problem 6 Draw a diagram like the one above, and check that the ratio of the corresponding sides of the triangles is 1 : 2 and that the corresponding angles are equal.

If, in this way, all the lines that pass through corresponding points of two figures meet at a single point O, and the ratio of the distances from O to corresponding points on each figure is constant, we say that those figures occupy a **position of similarity** and that O is the **center of similarity**.

<u>Two figures in a position of similarity are similar.</u>

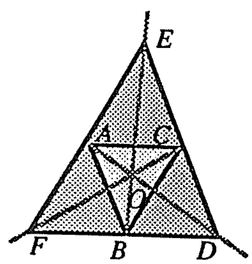
In each of the diagrams below $\Delta A'B'C'$ is similar to ΔABC, O is the center of similarity, and the similarity ratio is 1 : 2.

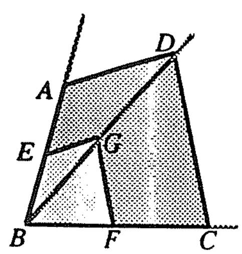


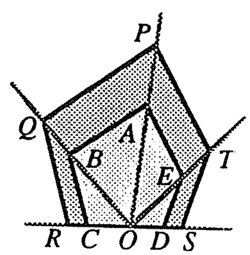


Problem 7 Figures (1)–(3) illustrate two polygons in a position of similarity. Give the centers of similarity, the corresponding vertices, and the corresponding sides. Express the similarity by using the symbol ~.

(1) triangle (2) quadrilateral (3) pentagon





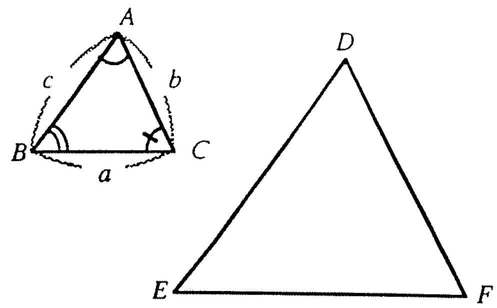


② Conditions for Similar Triangles

In the diagram below, assume that $\Delta ABC \sim \Delta DEF$, $BC = a$, $CA = b$, $AB = c$, and the similarity ratio for ΔABC and ΔDEF is 1 : 2.

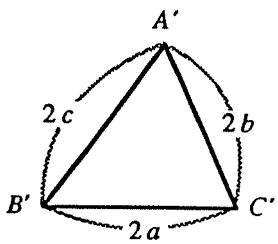
Problem 1 (1) Find the length of each side of ΔDEF.

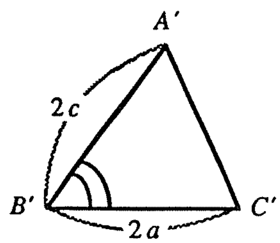
(2) Write an expression for the relation between the angles of ΔDEF and the angles of ΔABC.

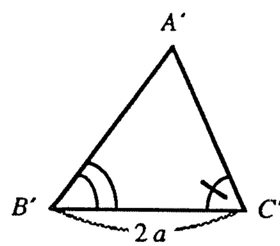


Apart from ΔDEF, if we draw $\Delta A'B'C'$ which satisfies conditions (1)–(3) below, then $\Delta A'B'C'$ is congruent to ΔDEF and similar to ΔABC.

(1)	(2)	(3)
$B'C' = 2a$	$B'C' = 2a$	$B'C' = 2a$
$C'A' = 2b$	$A'B' = 2c$	$\angle B' = \angle B$
$A'B' = 2c$	$\angle B' = \angle B$	$\angle C' = \angle C$







Problem 2 Explain why $\Delta A'B'C' \equiv \Delta DEF$ for conditions (1)–(3).

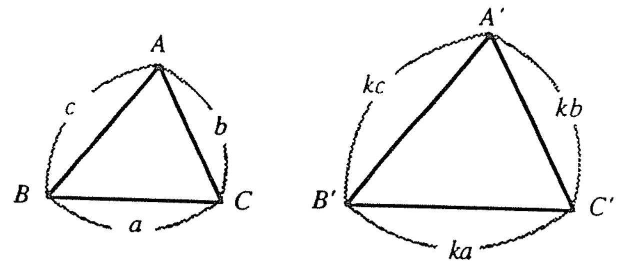
Like the conditions for congruent triangles, the following conditions for similar triangles are used to support proofs.

Conditions for Similar Triangles

Two triangles are similar if they satisfy any of the following three conditions.

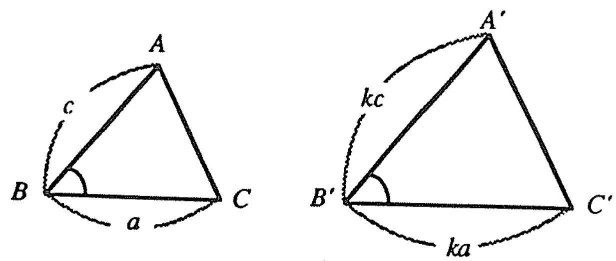
(1) The ratio of the lengths of all pairs of corresponding sides is the same.

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'}$$



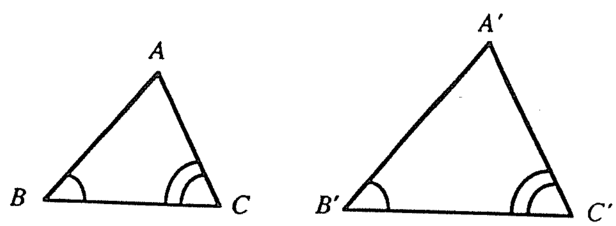
(2) The ratio of the lengths of two pairs of corresponding sides is the same, and the corresponding angles between them are equal.

$$\begin{cases} \dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} \\ \angle B = \angle B' \end{cases}$$

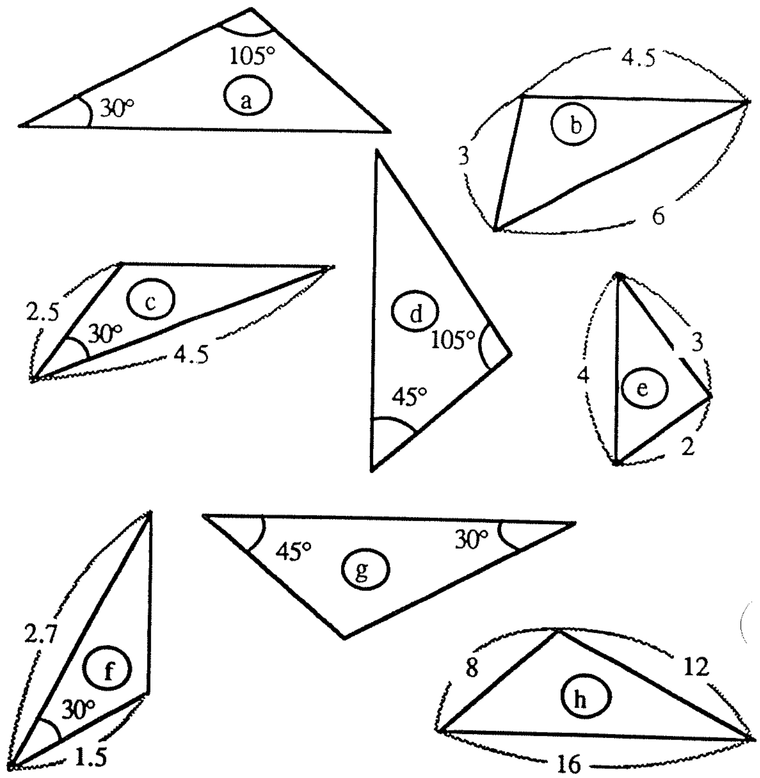


(3) Two pairs of corresponding angles are equal.

$$\begin{cases} \angle B = \angle B' \\ \angle C = \angle C' \end{cases}$$



Problem 3 Select the pairs of similar triangles from the figures below, and state which similarity conditions apply in each case.



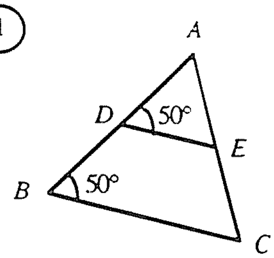
a, d, g
b, h, e
c, f

A

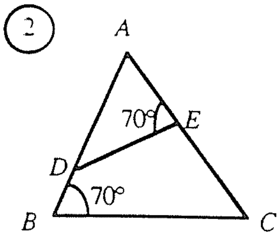
$\frac{a}{b} = \frac{a'}{b'}$

Problem 4 Write an expression for the pair of similar triangles in each figure below using the symbol ~. State which similarity condition applies in each case.

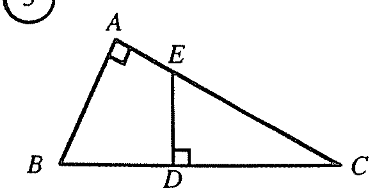
(1)



(2)



(3)



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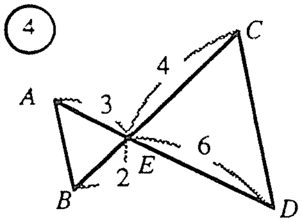
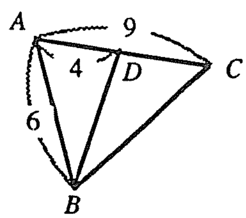
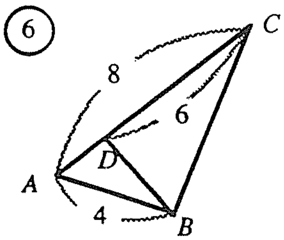


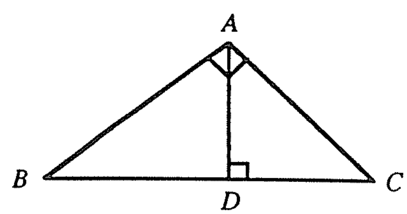
(5)



(6)



Example 1 In right triangle ABC, the right angle is $\angle A$, and a perpendicular drawn from A to BC intersects BC at D. Prove:



(1) $\triangle ABD \sim \triangle CBA$

(2) $\dfrac{AB}{CB} = \dfrac{BD}{BA}$

[Proof]

(1) In ΔABD and ΔCBA,

$$\begin{cases} \angle ADB = \angle CAB = \angle R \\ \angle B \text{ is common to both triangles.} \end{cases}$$

Since two pairs of corresponding angles are equal,

$$\Delta ABD \sim \Delta CBA$$

(2) Since ΔABD and ΔCBA are similar and the ratio of their corresponding sides is constant,

$$\frac{AB}{CB} = \frac{BD}{BA}$$

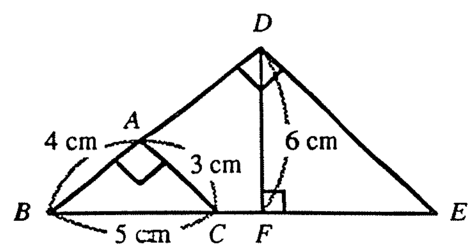
Problem 5 Starting from the hypothesis in Example 1, deduce

$$\frac{AD}{CD} = \frac{BD}{AD}$$

by following the same procedure as in the proof above.

Exercises

1. In the diagram to the right $\angle BDE$, $\angle BFD$, and $\angle BAC$ are all right angles.



   (1) Which triangle stands in a position of similarity to ΔBAC?

   (2) Which angle is equal to $\angle FDE$?

   (3) How long is DE?

2. Two right triangles are similar if one pair of corresponding acute angles is equal. Prove this.

3. Two isosceles triangles are similar if their vertices are equal. Prove this.

2 RATIOS AMONG PARALLEL LINES AND SEGMENTS

1 Ratio and Parallel Lines

The Basic Property of Ratios

Let's examine the basic property of ratios:

if $a : b = m : n$ then $an = bm$.

Problem 1 Find the numbers which go in the blanks, and check the above property.

(1) $2 : 3 = [\ \ ] : 6$

(2) $3 : [\ \ ] = 6 : 8$

Since $a : b = m : n$ is equivalent to

$$\frac{a}{b} = \frac{m}{n},$$

an
$a : b = m : n$
bm

we can multiply both sides of the equation by bn:

$$\frac{a}{b} \times bn = \frac{m}{n} \times bn$$

Hence, $$an = bm \quad (1)$$

Problem 2 Find the value of x in the following equations using the basic property of ratios.

(1) $x : 10 = 4 : 5$ (2) $2 : 3 = x : 5$

If we divide both sides of (1) by mn, we get

$$\frac{a}{m} = \frac{b}{n}$$

Thus, $a : m = b : n$

Therefore,

if $a : b = m : n$, then $a : m = b : n$

Problem 3 If $a : 5 = b : 3$, find the value of x in the following expression:

$$a : b = x : 3$$

Triangles and Ratio

Points D and E are located on sides AB and AC of ΔABC, such that

$$AD : DB = AE : EC$$

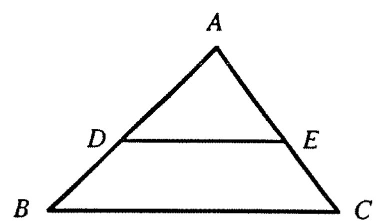
Let's consider what properties line segment DE, which connects D and E, will have.

Problem 4 Given $AD : DB = AE : EC = 3 : 2$.

(1) Find $AD : AB$ and $AE : AC$.

(2) Prove that $\Delta ADE \sim \Delta ABC$.

(3) Prove that $DE \parallel BC$.



Problem 5 As the converse of Problem 4, it is given that $DE \parallel BC$.

(1) Prove that $\Delta ADE \sim \Delta ABC$.

(2) Furthermore, assuming that $AD : DB = 3 : 2$, find

$AD : AB$, $AE : AC$, and $AE : EC$.

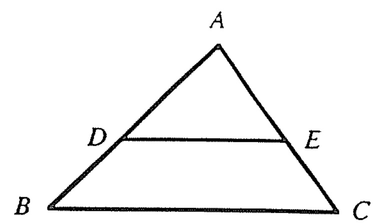
The following theorem holds for any ratio $AD : DB$.

Triangles and Ratio

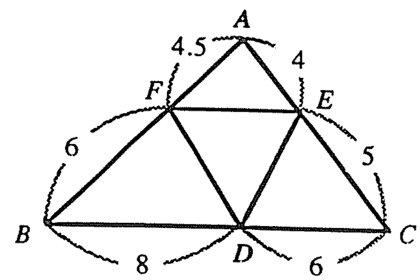
Theorem: Given points D and E on sides AB and AC of ΔABC,

(1) If $\frac{AD}{DB} = \frac{AE}{EC}$, then $DE \parallel BC$.

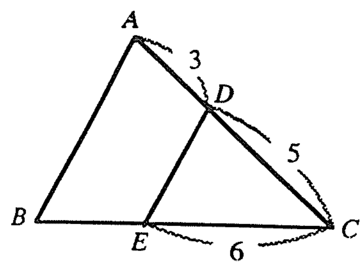
(2) If $DE \parallel BC$, then $\frac{AD}{DB} = \frac{AE}{EC}$.



Problem 6 In the diagram to the right, which of the segments DE, EF, and FD is parallel to a side of ΔABC?

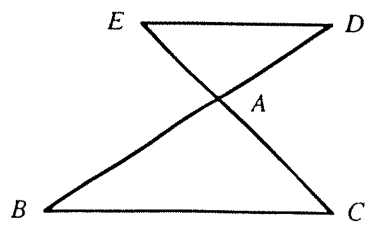


Problem 7 In the diagram to the right, if $DE \parallel AB$, find the length of BE.



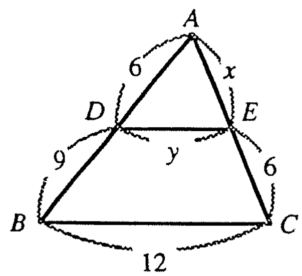
If points D and E lie on the extensions of sides BA and CA of ΔABC, and if $DE \,//\, BC$, then ΔADE and ΔABC must be similar.

Therefore, $\frac{AD}{AB} = \frac{AE}{AC}$.

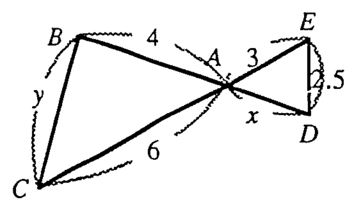


Problem 8 Given that $DE \,//\, BC$ in each of the diagrams below, find the lengths of x and y.

(1)



(2)



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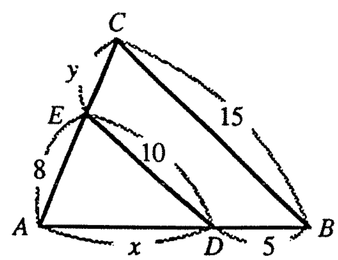
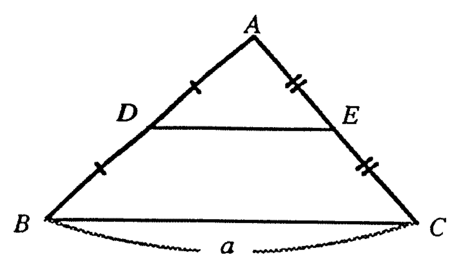


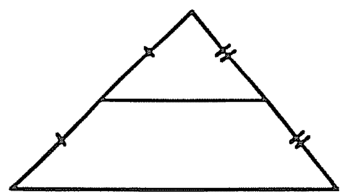
The Midpoint Connection Theorem

Problem 9 Assume that points D and E are the midpoints of sides AB and AC of ΔABC, respectively, and the length of side BC is a. What is the length of segment DE?

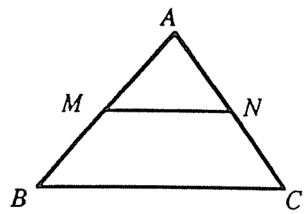


The Midpoint Connection Theorem

Theorem: A line segment connecting the midpoints of two sides of a triangle is parallel to the third side, and it is half as long as the third side.



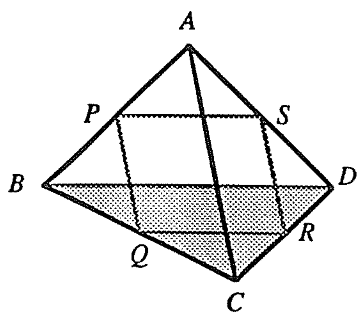
Problem 10 Write expressions for the hypothesis and conclusion of the midpoint connection theorem in terms of the figure at the right.



Hypothesis: ☐

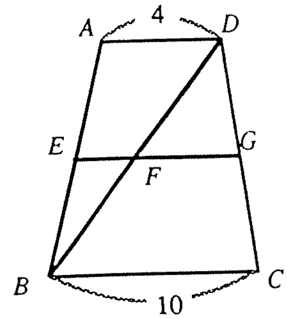
Conclusion: ☐

Problem 11 The diagram to the right shows a regular tetrahedron $ABCD$ with an edge of 6 cm. If we cut this regular tetrahedron with a plane passing through the midpoints of edges AB, BC, CD, and DA, what is the perimeter of the cut surface?



Problem 12 If the midpoints of sides BC, CA, and AB of ΔABC are D, E, and F, respectively, then $\Delta DEF \sim \Delta ABC$. Prove this.

Problem 13 The diagram to the right shows trapezoid $ABCD$, where $AD \parallel BC$. We assume that E is the midpoint of side AB, and that a line drawn from E parallel to side BC intersects BD and CD at F and G, respectively. Find the lengths of EF and EG

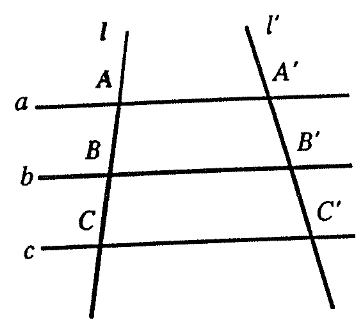


Parallel Lines and Ratio

Parallel Lines and Ratio

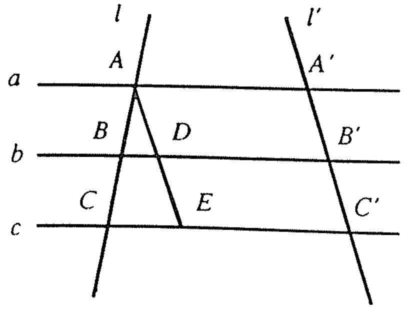
Theorem: If three parallel lines a, b, and c intersect line l at A, B, and C, respectively, and intersect line l' at A', B', and C', then

$$\frac{AB}{BC} = \frac{A'B'}{B'C'}.$$



[Proof]

We can draw a straight line through A parallel to l' and label the points at which it intersects b and c as D and E, respectively. In $\triangle ACE$, since



$$BD \parallel CE,$$

we know that

$$\frac{AB}{BC} = \frac{AD}{DE} \qquad (1)$$

Since quadrilaterals $ADB'A'$ and $DEC'B'$ are both parallelograms, we also know that

$$AD = A'B',\ DE = B'C' \qquad (2)$$

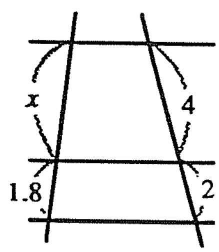
From (1) and (2),

$$\frac{AB}{BC} = \frac{A'B'}{B'C'}$$

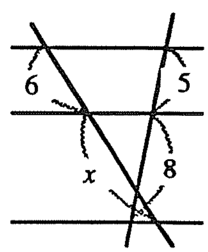
In this proof, we considered the case in which l and l' are not parallel, and A and A' are not the same point, but the proof would be much simpler if l and l' were parallel, or if A and A' were the same point.

Problem 14 In each of the diagrams below, two straight lines intersect three parallel lines. Find the length of x in each diagram.

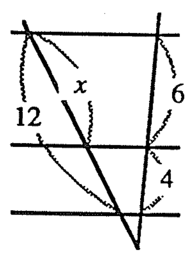
(1)



(2)



(3)



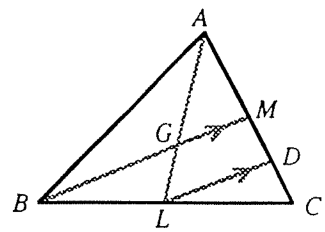
② The Centroid of a Triangle

A line segment that connects a vertex of a triangle with the midpoint of the opposite side is called a **median**.

Problem 1 Draw a triangle, and then draw all its medians.

If two medians AL and BM of ΔABC intersect at point G, then

$$AG : GL = 2 : 1$$



Problem 2 Assume that a straight line passing through L parallel to BM intersects AC at D. Now prove

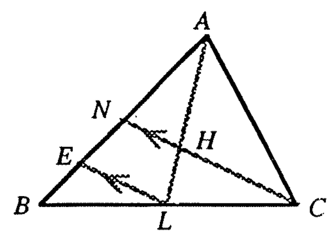
(1) $MD = DC$ (2) $AM : MD = 2 : 1$

and use those facts to demonstrate that $AG : GL = 2 : 1$.

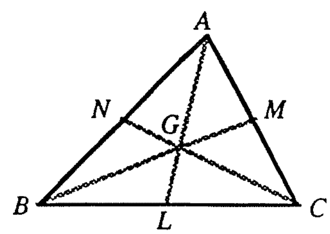
Next, if we assume that two medians AL and CN of ΔABC intersect at H, we can show that

$$AH : HL = 2 : 1$$

by the same procedure as in Problem 2.



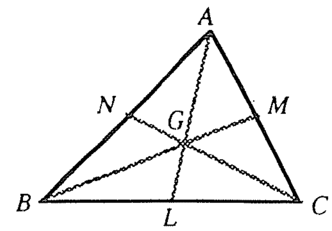
Since both G and H divide median AL into parts with a ratio of $2 : 1$, G and H must coincide. In other words, the three medians AL, BM, and CN all intersect at point G, which divides median AL into parts with a ratio of $2 : 1$.



Centroid Theorem

Theorem: The three medians of a triangle intersect at a single point. That intersection divides the medians into parts with a ratio of $2 : 1$.

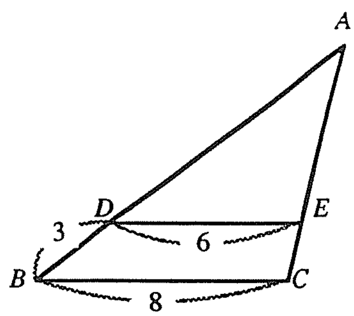
The point at which the three medians of a triangle intersect is called the **centroid** of that triangle. The centroid of a triangle is the point that divides the medians into parts with a ratio of 2 : 1.

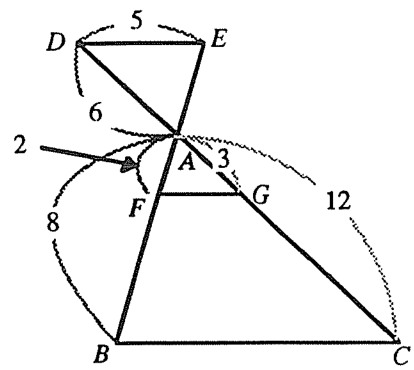


Exercises

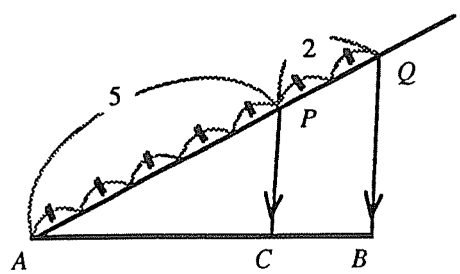
1. (1) If $DE \parallel BC$ in the diagram at the left below, how long is AD?

   (2) Assume $BC \parallel DE$ in the diagram at the right below. Prove that $FG \parallel BC$, and find the lengths of AE, BC, and FG.





2. The diagram at the right shows how to find a point C that divides segment AB into parts with a ratio of 5 : 2. Find the point that divides AB into parts with a ratio of 3 : 2 in the same way.

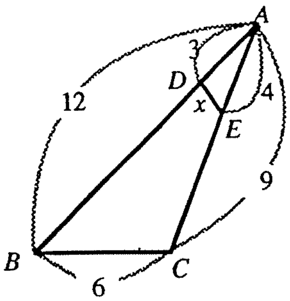


Chapter Exercises

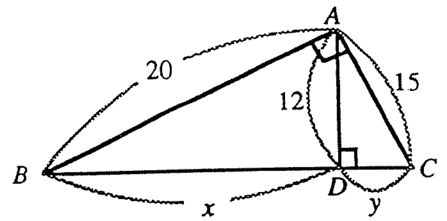
A

1. Find the length of x in diagram (1) and x and y in diagram (2) below.

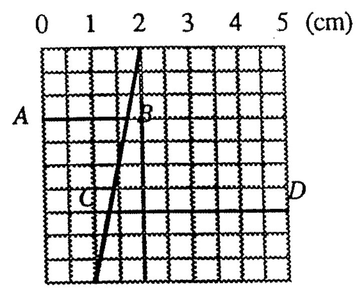
(1)



(2)

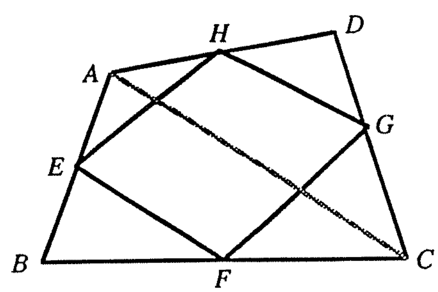


2. Segment AB in the diagram to the right is 1.7 cm long. Think about why this is so. With this in mind, draw a segment 1.4 cm long on the diagram. How long is CD?



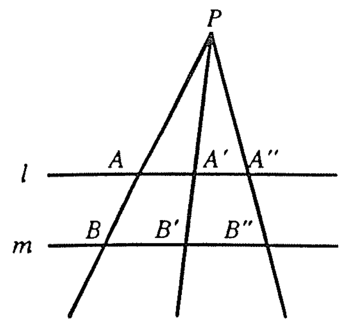
3. In ΔABC, if D is the midpoint of AB, and a straight line passing through B parallel to DC intersects the extension of AC at point E, then $AC = CE$. Prove this.

4. In the diagram to the right, if we connect the midpoints of each side of quadrilateral $ABCD$ in succession to form quadrilateral $EFGH$, the new figure must be a parallelogram. Prove this.



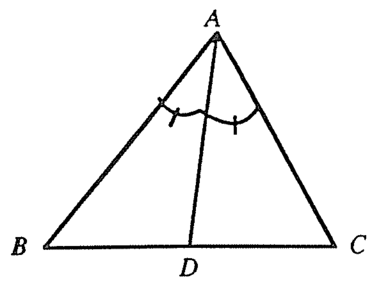
5. If the midpoints of two sides BC and CD of $\square ABCD$ are E and F, respectively, and BD intersects AE and AF at P and Q, respectively, then $BP = PQ = QD$. Prove this.

1. As shown in the diagram to the right, parallel lines l and m intersect three straight lines passing through point P at A, A', A'', and B, B', B'', respectively. Prove that $\frac{AA'}{A'A''} = \frac{BB'}{B'B''}$.



2. If the bisector of $\angle A$ in $\triangle ABC$ intersects side BC at D, then use the following procedure to prove that

$$\frac{AB}{AC} = \frac{BD}{DC}.$$

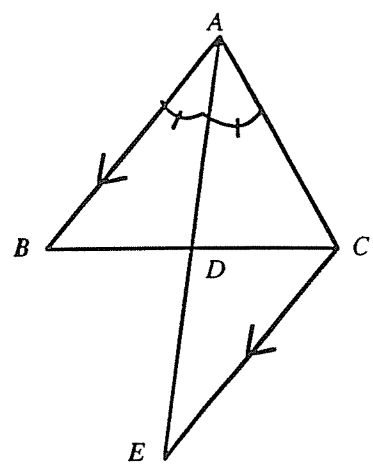


First we draw a straight line through point C parallel to AB so that it intersects the extension of AD at point E. Then we can prove that

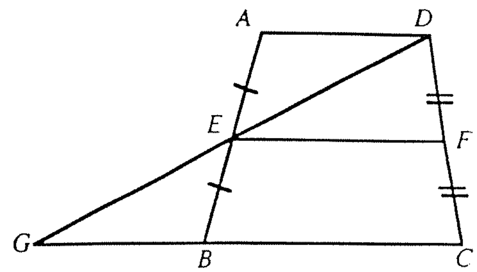
(1) $AC = CE$

(2) $\frac{AB}{CE} = \frac{BD}{DC}$

Now we can demonstrate the desired conclusion based on (1) and (2).



3. Let E and F be the midpoints of sides AB and CD of trapezoid $ABCD$ with bases AD and BC. If the extension of DE intersects the extension of CB at point G, then



(1) $DE = EG$

(2) $EF \parallel BC$

Prove these conclusions.