J8-1

2 CONGRUENT FIGURES

1 Congruent Figures

Moving a figure from one position to another without changing its shape and size is called a Euclidean transformation.

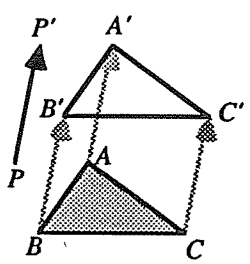
There are three distinct types of Euclidean transformations:

(1) Moving a figure so that all the points on it move a fixed distance in a fixed direction.

(2) Rotating a figure about a fixed point as its center of rotation.

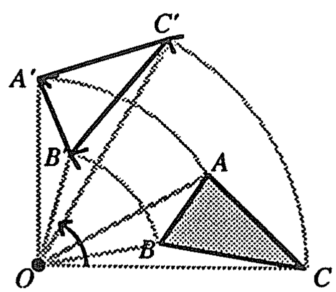
(3) Flipping a figure over a fixed straight line as its axis.

(1)



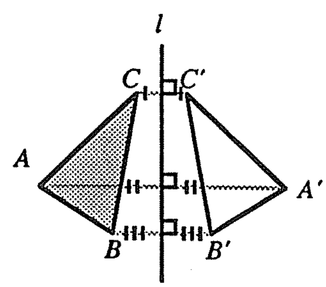
translation

(2)



rotation

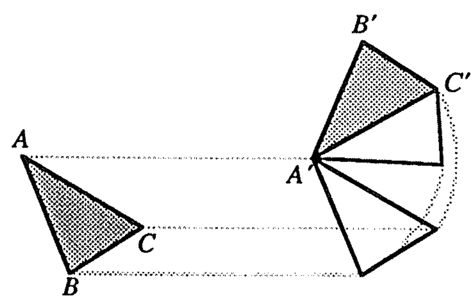
(3)



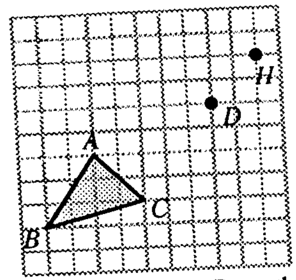
reflection

These three transformations are called **translation, rotation,** and **reflection,** respectively. Any Euclidean transformation of a plane figure can be accomplished by combining these three types of transformations.

If there are two figures on a plane, and we can lay one on top of the other by moving it, these two figures are said to be congruent.



Problem 1 Use the graph paper diagram to the right to do the following problems.



(1) Draw ΔDEF by translating ΔABC so that vertex A of ΔABC lies on point D.

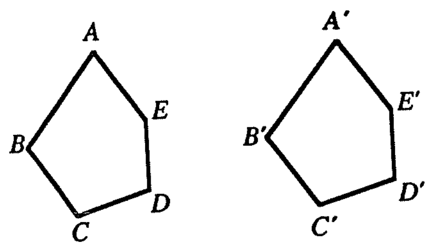
(2) Draw ΔDGH by rotating ΔDEF about vertex D so that one of its vertices moves to H.

(3) Draw ΔDIH by reflecting ΔDGH about straight line DH.

(4) Give the corresponding vertices, sides, and angles in ΔABC and ΔDIH.

Property of Congruent Figures

Corresponding angles and line segments in congruent figures are equal.



If pentagon $ABCDE$ and pentagon $A'B'C'D'E'$ in the diagram above are congruent, and the corresponding vertices are A and A', B and B', C and C', D and D', and E and E', we can write

$$\textbf{pentagon } ABCDE \equiv \textbf{pentagon } A'B'C'D'E'$$

The symbol "≡" means "is congruent to." When using this symbol, we write the names of corresponding vertices consecutively, as they occur along the perimeter of the figures.

Since corresponding angles and line segments in congruent figures are equal, as in

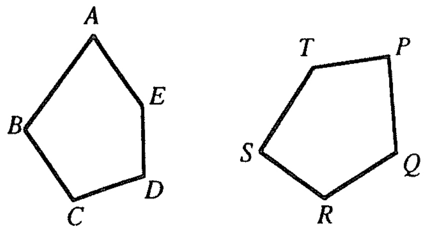
$$\text{pentagon } ABCDE \equiv \text{pentagon } A'B'C'D'E'$$

we can conclude that

$$\angle A = \angle A',\ \angle B = \angle B',\ \angle C = \angle C',\ \angle D = \angle D',\ \angle E = \angle E'$$

$$AB = A'B',\ BC = B'C',\ CD = C'D',\ DE = D'E',\ EA = E'A'$$

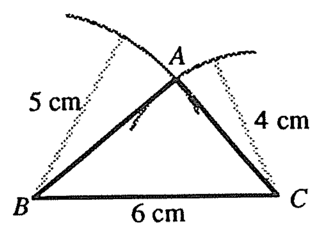
Problem 2 If pentagon $ABCDE \equiv$ pentagon $PQRST$, identify their corresponding sides and angles.



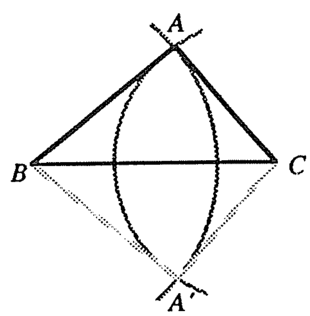
Example 3 If $\Delta ABC \equiv \Delta DEF$, find the equal sides and equal angles of these two triangles and write expressions equating them.

2 Conditions for Congruent Triangles

The diagram to the right shows how to draw ΔABC, where $AB = 5$ cm and $AC = 4$ cm, if we have determined that line segment BC is 6 cm long.



When we determine the position of line segment BC, once we decide on which side of BC to draw ΔABC, we have then effectively determined the position of point A, which is 5 cm from point B, which is in turn 4 cm from point C. Thus, we can draw only one ΔABC.



In general, once we have determined the position of one side of a triangle and decided on which side of that side we will put the triangle, the position of the triangle is effectively determined if we know any of the following pieces of information:

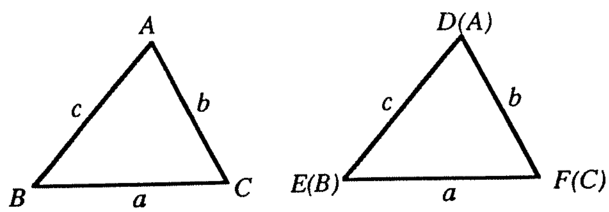
(1) the lengths of three sides;

(2) the lengths of two sides and the measure of the angle between them;

(3) the length of one side and the measures of the angles at both ends of that side.

Conditions for Congruent Triangles

Let's assume that the three sets of corresponding sides in ΔABC and ΔDEF in the diagram to the right are of equal length.



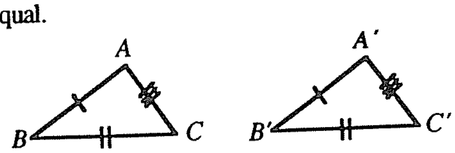
Now, let's move ΔABC, placing BC on EF and vertex A on the same side of EF as vertex D. As we have just learned, if the lengths of the three sides are determined, the position of the triangle is uniquely determined, and these two triangles match each other perfectly. Therefore, if all three pairs of corresponding sides in two triangles are equal, those two triangles must be congruent.

Here we have applied condition (1) from the preceding page. If we apply conditions (2) and (3) in the same way, we can establish the following conditions for congruent triangles.

Conditions for Congruent Triangles

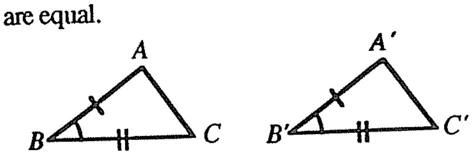
Two triangles are congruent if any one of the following conditions is satisfied.

(1) Each of the three pairs of corresponding sides is equal.



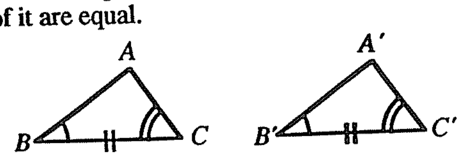
$$\begin{cases} AB = A'B' \\ BC = B'C' \\ CA = C'A' \end{cases}$$

(2) Two corresponding sides and the angle between them are equal.



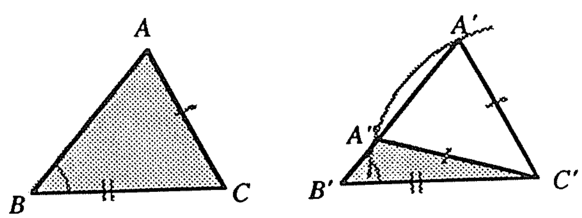
$$\begin{cases} AB = A'B' \\ BC = B'C' \\ \angle B = \angle B' \end{cases}$$

(3) One corresponding side and the angles at both ends of it are equal.



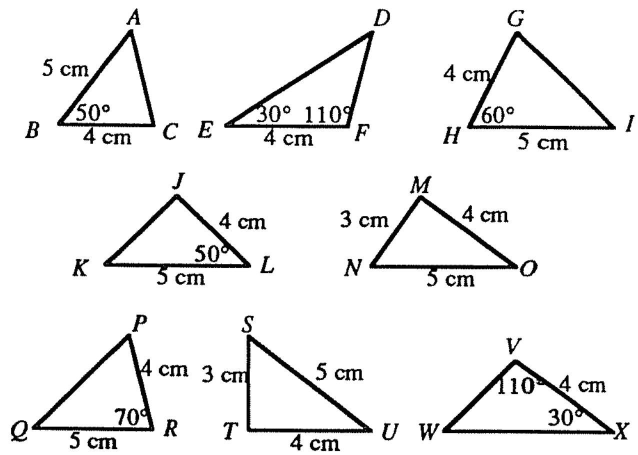
$$\begin{cases} AB = A'B' \\ \angle B = \angle B' \\ \angle C = \angle C' \end{cases}$$

It is possible for two triangles not to be congruent even though two corresponding sides and one corresponding angle are equal (as in ΔABC and $\Delta A'B'C'$ at the right), if the equal angles do not lie between the two equal sides.



When we try to determine whether or not two triangles are congruent, we need not place one on top of the other; rather, we can judge whether they are congruent by whether they satisfy any of the conditions for congruence.

Problem 1 Which triangles below are congruent with other triangles pictured? Write appropriate expressions using the symbol ≡. Tell which congruence condition is satisfied for each pair.



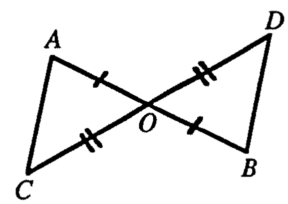
Problem 2 In Problem 1, you determined which triangle is congruent with ΔABC. Which of its sides is equal to side AC? Which angle is equal to $\angle A$?

We can use the necessary conditions for congruent triangles to demonstrate the properties of various figures.

If point O is the midpoint of both AB and CD in the diagram to the right, then

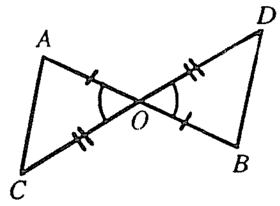
$$AC \parallel BD$$

In order to prove this, it suffices to show that one pair of alternate interior angles, $\angle A$ and $\angle B$, for example, is equal.



In ΔAOC and ΔBOD,

$$\begin{cases} AO = BO \\ CO = DO \\ \angle AOC = \angle BOD \end{cases}$$



Therefore, because two sides and the intervening angle of ΔAOC and ΔBOD are equal,

$$\Delta AOC = \Delta BOD$$

Then, since $\angle A$ and $\angle B$ are corresponding angles,

$$\angle A = \angle B$$

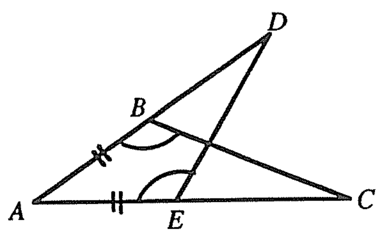
Since $\angle A$ and $\angle B$ are equal,

$$AC \parallel BD$$

Problem 3 In the diagram to the right, if we assume that $\angle ABC = \angle AED$ and $AB = AE$, then

$$BC = ED$$

Show why this is true.

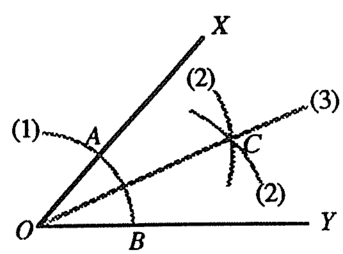


We can use the necessary conditions for congruent triangles to show the validity of figures we construct.

Given $\angle XOY$, we could draw its bisector OC in the following way.

(1) Draw a circle with an arbitrary radius around center O, and label its intersections with sides OX and OY as A and B, respectively.

(2) Taking A and B as their centers, we draw circles of equal radius and label the intersection of those circles C, as in the diagram.

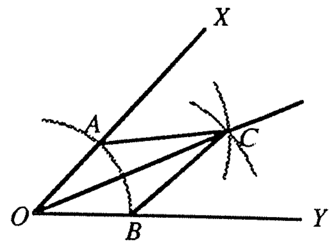


(3) Now draw ray OC, which is the bisector of $\angle XOY$.

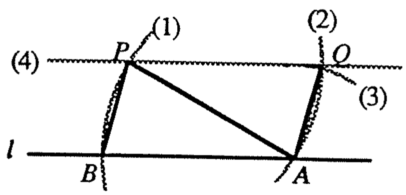
In order to demonstrate that OC is the bisector of $\angle XOY$, it is sufficient to show that $\angle AOC = \angle BOC$.

Problem 4 Take $\angle XOY$ above, and construct two triangles by drawing lines from A to C and B to C. Show why these triangles are congruent, and then demonstrate that

$$\angle AOC = \angle BOC$$

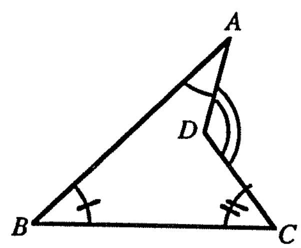


Problem 5 Given straight line l and point P not on line l, we can draw a straight line through P parallel to l, as in the diagram to the right, by finding points A, B, and Q such that $AB = AP = PQ$ and $BP = AQ$. Show that $PQ \parallel l$.



③ Demonstrating the Properties of Figures

Problem 1 In your notebook draw a diagram like the one to the right, and measure $\angle A$, $\angle B$, $\angle C$, and $\angle ADC$ using a protractor. What relation holds among these angles?



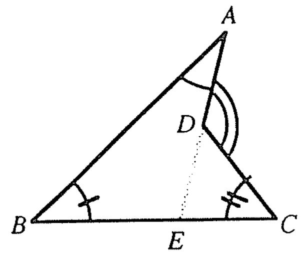
From our observations in Problem 1, we can predict that

$$\angle ADC = \angle A + \angle B + \angle C$$

The reason this always holds true can be demonstrated in the following way.

Let us extend AD so that it intersects BC at E. Since $\angle DEC$ is an exterior angle of $\triangle ABE$,

$$\angle DEC = \angle A + \angle B \qquad (1)$$



Also, since $\angle ADC$ is an exterior angle of $\triangle DEC$,

$$\angle ADC = \angle DEC + \angle C \qquad (2)$$

From (1) and (2),

$$\angle ADC = \angle A + \angle B + \angle C \qquad (3)$$

We have now verified that our prediction was correct.

Here we demonstrated that (3) holds true by utilizing the fact that an exterior angle of a triangle is equal to the sum of the non-adjacent interior angles.

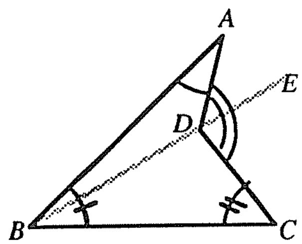
A proper explanation of why a certain fact is true, without relying on experiments or intuition but rather on previously established properties, is called a **proof**.

An explanation like the one on pages 118–119 using the conditions for congruent triangles is also a proof.

Problem 2 Prove that

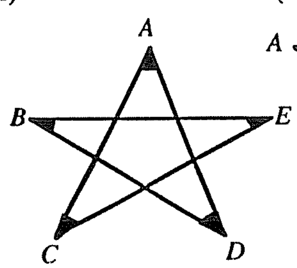
$$ADC = \angle A + \angle B + \angle C$$

in the diagram to the right, assuming that DE is an extension of BD. Think about other ways of proving this.

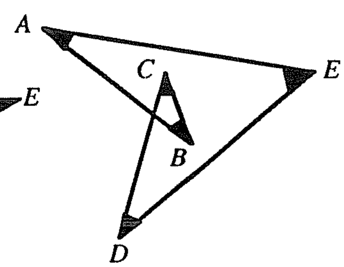


Problem 3 In diagrams (1) and (2) below, the sum of the five marked angles is 180°. Prove this from what we learned on the preceding page.

(1)



(2)



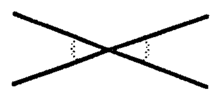
Now let's start proving various properties of figures.

Fundamental Properties

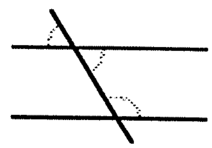
In order to be able to prove properties of figures, we must first explicitly state fundamental properties we can use to support our proofs. Let's summarize here the main fundamental properties we have learned so far, which will be used very often from now on.

Fundamental Properties

- Vertical angles are equal.



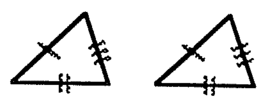
- If one straight line intersects two other lines,
  - if the two other lines are parallel, then the corresponding angles and the alternate interior angles are equal.
  - if the corresponding angles or the alternate interior angles are equal, then the two lines are parallel.



- The sum of the exterior angles of a triangle is equal to the sum of the two non-adjacent interior angles.

- The sum of the interior angles of a triangle is $2\angle R$.
- Conditions for congruent triangles:

(1) All three corresponding sides are equal.



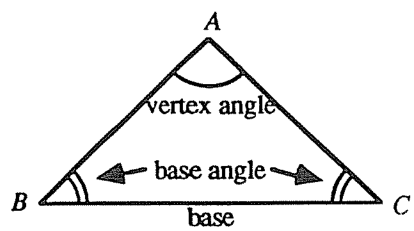
(2) Two corresponding sides and the angle between them are equal.

(3) One corresponding side and the two angles at each end of it are equal.

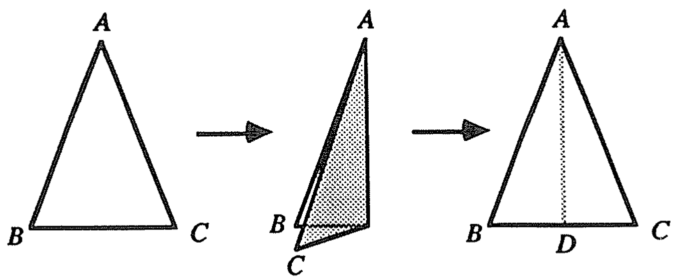
3 PROPERTIES OF TRIANGLES

1 Properties of Isosceles Triangles

A triangle in which two sides are of equal length is called an isosceles triangle. In an isoceles triangle, the angle between the two equal sides is called the **vertex angle,** the side facing the vertex angle is the **base,** and the angles at the ends of the base are the **base angles.**



Problem 1 If we fold isosceles triangle ABC, where $AB = AC$, so that sides and AC lie together, what happens to $\angle B$ and $\angle C$?



Isosceles triangles have the following property:

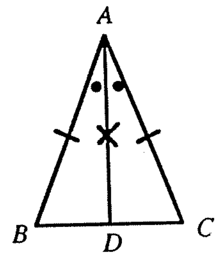
The base angles of an isosceles triangle are equal.

Problem 1 is an attempt to check this property by experiment.

Next, let's give a formal proof of this property.

Problem 2 What does folding the triangle in Problem 1 do to $\angle A$?

In isosceles triangle ABC, where $AB = AC$, let's assume that the bisector of $\angle A$ intersects BC at D.



In ΔABC and ΔACD,

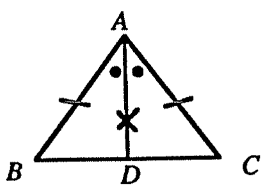
$$\begin{cases} AB = AC \\ AD = AD \\ \angle BAD = \angle CAD \end{cases}$$

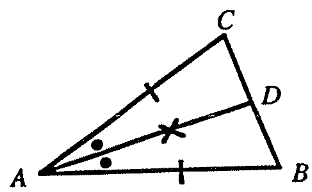
Since the two corresponding sides and the angle between them are equal,

$$\Delta ABD \equiv \Delta ACD$$

Hence, $\angle B = \angle C$

This proof holds for any ΔABC where $AB = AC$.





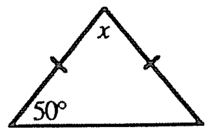
Thus, we can state that the two base angles are equal in any isosceles triangle whatsoever. This property will later be used often to support proofs of the properties of figures. A fact which has been formally proved and is used often is called a theorem.

Property of the Base Angles of an Isosceles Triangle

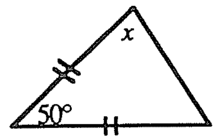
Theorem: The base angles of an isosceles triangle are equal.

Problem 3 In the diagrams below, assume that sides with the same marks are equal, and find the measure of $\angle x$.

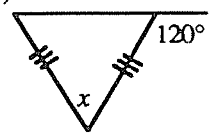
(1)



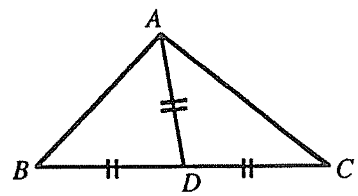
(2)



(3)



Problem 4 In the diagram to the right, if



$$DA = DB = DC,$$

then

$$\angle BAC = \angle R$$

Prove this.

A statement that explicitly gives the meaning of a term is called a **definition.** For example, the following statement is a definition:

An isosceles triangle is a triangle in which two sides are of equal length.

The following is the definition of an equilateral triangle:

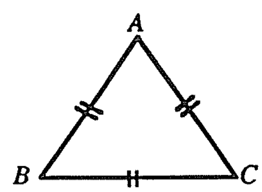
An equilateral triangle is a triangle with three equal sides.

Using the property of the base angles of an isosceles triangle given on the preceding page, let's prove that

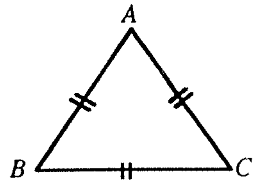
<u>the three angles of an equilateral triangle are equal.</u>

[Approach] In ΔABC, assuming that $AB = BC = CA$, it is sufficient to prove that

$$\angle A = \angle B = \angle C$$



[**Proof**] $\triangle ABC$ can be regarded as an isosceles triangle where $AB = AC$, and so by the theorem on page 125,



$$\angle B = \angle C \qquad (1)$$

Since $\triangle ABC$ can also be regarded as an isosceles triangle where $BA = BC$,

$$\angle A = \angle C \qquad (2)$$

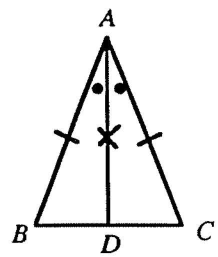
From (1) and (2),

$$\angle A = \angle B = \angle C$$

An isosceles triangle also has the following property:

The bisector of the vertex angle of an isosceles triangle is a perpendicular bisector of the base.

Thus, in the diagram to the right, if



$$AB = AC, \angle BAD = \angle CAD,$$

then

$$BD = CD, AD \perp BC$$

Problem 5 Prove that $\triangle ABD \equiv \triangle ACD$, and show that $BD = CD$.

Let's prove that $AD \perp BC$.

Since	$\triangle ABD$	$\equiv \triangle ACD$,	
we have	$\angle ADB$	$= \angle ADC$	(1)
However,	$\angle ADB + \angle ADC$	$= 2\angle R$	(2)
From (1) and (2),	$2\angle ADB$	$= 2\angle R$	
Hence,	$\angle ADB$	$= \angle R$	
Thus,	AD	$\perp BC$	

② Congruent Right Triangles

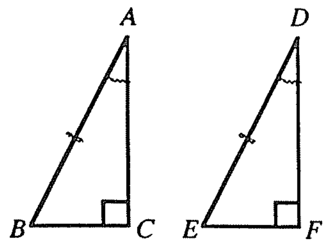
We have learned to use the conditions for congruent triangles to examine the properties of figures. Now let's consider the conditions for congruent right triangles.

Problem 1 In ΔABC and ΔDEF, if

$$\begin{cases} \angle C = \angle F = \angle R \\ AB = DE \\ \angle A = \angle D \end{cases}$$

then

$$\Delta ABC \equiv \Delta DEF$$

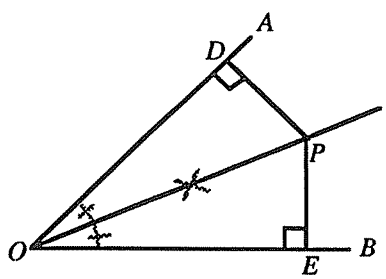


Prove this using the necessary conditions for congruent triangles.

As Problem 1 shows, two right triangles are congruent if the hypotenuse and one acute angle in each figure are equal.

Problem 2 Given that P is a point on the bisector of $\angle AOB$, we can draw perpendiculars from P to intersect OA and OB at D and E, respectively. Then

$$PD = PE$$



Prove this.

Two right triangles must also be congruent if both the hypotenuse and one other side are equal.

Thus, in $\triangle ABC$ and $\triangle DEF$, if

$$\begin{cases} \angle C = \angle F = \angle R \\ AB = DE \\ AC = DF \end{cases}$$

then

$$\triangle ABC \equiv \triangle DEF$$

Let's prove this.

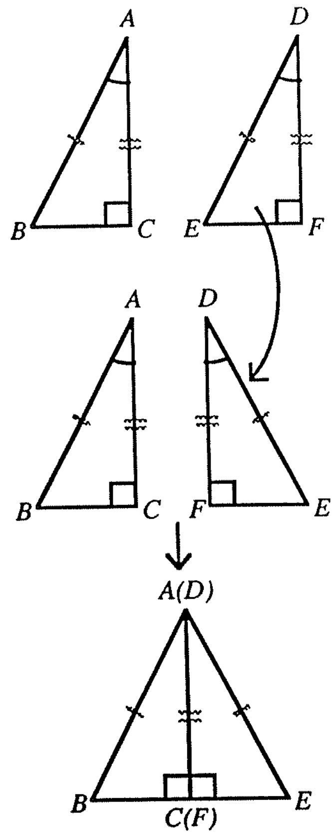
Since $DF = AC$, as we see in the diagram to the right, we can reflect $\triangle DEF$ and lay DF on AC, thus conjoining $\triangle DEF$ and $\triangle ABC$. Then, since

$$\angle C = \angle F = \angle R$$

we know that

$$\angle BCE = 2\angle R$$

Thus, line segments BC and CE have been joined into one line segment. Therefore, $\angle B$ of $\triangle ABC$ and $\angle E$ of $\triangle DEF$ are now two angles $\angle ABE$ and $\angle AEB$ of a single triangle ABE.



Problem 3 In $\triangle ABE$ which we have just created,

$$\angle B = \angle E$$

Show why this is so. Use this fact to prove that

$$\triangle ABC \equiv \triangle DEF$$

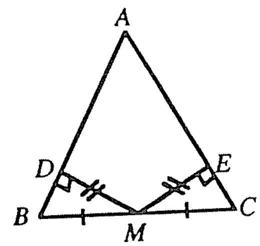
Conditions for Congruent Right Triangles

Theorem: Two right triangle are congruent if either of the following two conditions is satisfied.

(1) The hypotenuse and one acute angle of one triangle are equal to the corresponding hypotenuse and acute angle of the other.

(2) The hypotenuse and one other side of one triangle are equal to the corresponding hypotenuse and one other side of the other triangle.

Problem 4 Let's draw perpendiculars from midpoint M of side BC of ΔABC to sides AB and AC such that they intersect sides AB and AC at D and E, respectively. If $MD = ME$, then $\angle B = \angle C$. Prove this.



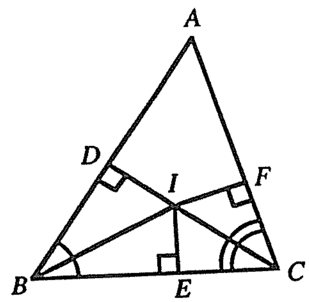
The Intersection of the Bisectors of the Interior Angles of a Triangle

Let's use the conditions for congruent right triangles to examine the properties of the bisectors of the interior angles of a triangle.

Problem 5 In the diagram to the right, the bisectors of $\angle B$ and $\angle C$ in ΔABC intersect at I. We have drawn perpendiculars from I to intersect the three sides $AB, BC,$ and CA at $D, E,$ and F, respectively.

(1) Prove that $ID = IE = IF$.

(2) Prove that ray AI bisects $\angle BAC$.



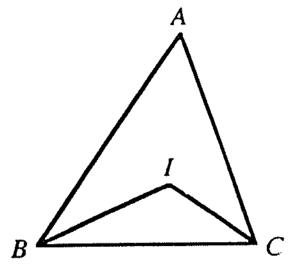
Problem 5 enables us to make the following generalization.

The bisectors of the three interior angles of a triangle intersect at one point equidistant from the three sides.

Problem 6 In the diagram to the right, point I is the intersection of the bisectors of the three sides of ΔABC.

(1) If $\angle A = 62°$, what is the measure of $\angle BIC$?

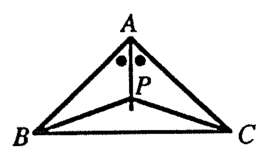
(2) Assuming that $\angle A = a°$, express $\angle BIC$ in terms of a.



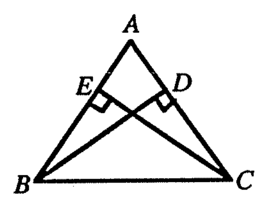
Exercises

1. What is the measure of the base angles of an isosceles triangle with a vertex angle of 60°?

2. Given that P is a point on the bisector of vertex angle $\angle A$ of isosceles triangle ABC, then $PB = PC$. Prove this.



3. If we draw perpendiculars from the endpoints B and C of base BC of isosceles triangle ABC to intersect sides AC and AB at D and E, respectively, then $BD = CE$. Prove this.

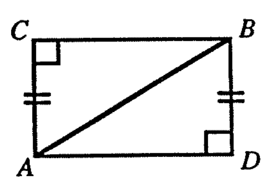


4. In the diagram to the right, if

$$AC = BD \text{ and } \angle C = \angle D = \angle R,$$

then $\quad BC = AD$ and $BC \parallel AD$.

Prove this.



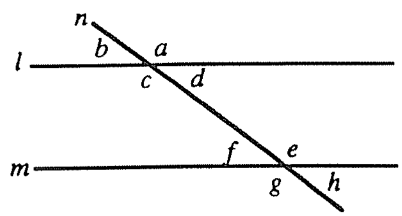
Chapter Exercises

A

1. In the diagram to the right, if

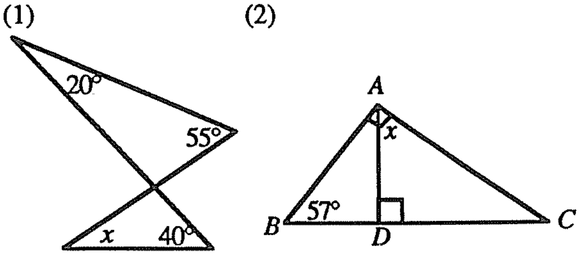
$l \parallel m$ and $\angle d = 27°$

find the measure of the other angles.

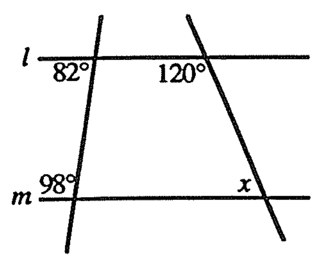


2. Find the measure of $\angle x$ in the diagrams below.

(1) (2)



3. We can see that $l \parallel m$ from the angles given in the diagram to the right. Explain why this is true. Find the measure of $\angle x$.

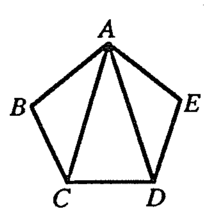


4. If $ABCDE$ in the diagram to the right is a regular pentagon, find the measure of the following angles:

(1) $\angle BAC$

(2) $\angle CAD$

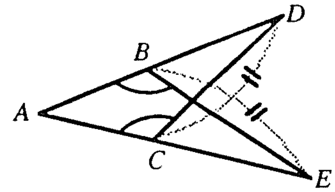
(3) $\angle ACD$



5. In the diagram to the right, if

$\angle ABE = \angle ACD$ and $BE = CD$

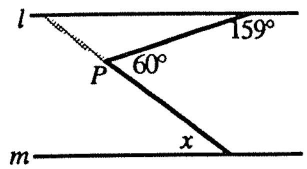
then $BD = CE$. Prove this.



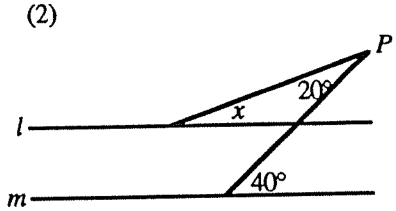
B

1. In the diagrams below, $l \parallel m$. Find the measure of $\angle x$.

(1)



(2)



2. Answer the following questions.

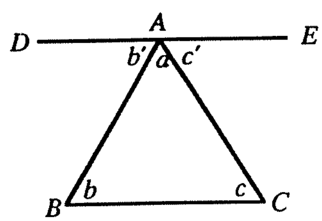
(1) How many sides does a regular polygon have if its exterior angles are 45°?

(2) What is the sum of the interior angles of a twenty-sided figure?

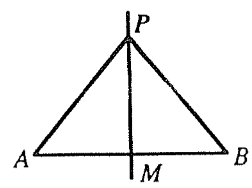
(3) How many sides does a polygon have if the sum of its interior angles is 1260°?

3. In isosceles triangle ABC, where $AB = AC$, the bisector of $\angle B$ and AC intersect at point D. If $DA = BD$, find the measure of $\angle A$.

4. Straight line DE passes through vertex A of ΔABC parallel to side BC. Which angles make up pairs of alternate interior angle with $\angle b$, $\angle c$? Use the answer to this question to prove that the sum of the interior angles of a triangle is $2\angle R$.



5. If P is a point on the perpendicular bisector of line segment AB, then $AP = BP$. Prove this using the necessary conditions for congruent triangles.



6. Point P is equidistant from the endpoints A and B of line segment AB and lies on its perpendicular bisector. Prove this by drawing a line from the midpoint M of AB to P and using the necessary conditions for congruent triangles.