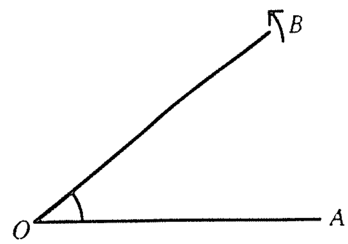
CHAP 5 [J8-04-01-11]

1 ANGLES AND PARALLEL LINES

Angles



In the figure to the right, if we fix the position of ray OA on one plane and rotate ray OB from OA in the direction indicated by the arrow, $\angle AOB$ gradually widens. The angle resulting from rotating the ray completely is 360°, and $\frac{1}{4}$ of that angle is a right angle, which we write as $\angle R$. We can express the measure of an angle using a right angle as the basic unit.

$90° = \text{right angle} = \angle R$ $\quad$ $180° = 2 \text{ right angles} = 2\angle R$

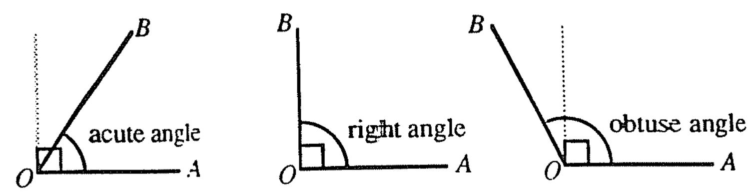
$360° = 4 \text{ right angles} = 4\angle R$ $\quad$ $45° = \frac{1}{2} \text{ right angle} = \frac{1}{2}\angle R$

Problem 1 Express the following angles in terms of $\angle R$:

(1) 270° $\quad$ (2) 60°

(3) 150°

An angle greater than 0° but less than 90° is called an **acute angle**, and an angle greater than 90° but less than 180° is called an **obtuse angle**.



$0° < \angle AOB < 90°$ $\quad$ $\angle AOB = 90°$ $\quad$ $90° < \angle AOB < 180°$

Problem 2 Which of the following angles are acute? Which are obtuse?

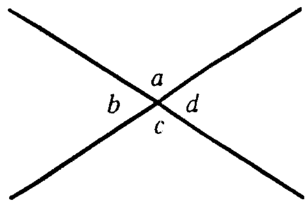
(1) $35°$ (2) $90°$

(3) $130°$ (4) $\frac{3}{2}\angle R$

(5) $\frac{3}{4}\angle R$ (6) $206°$

Vertical Angles

Two intersecting straight lines form angles at the intersection. Angles facing each other at the intersection of two lines, such as $\angle a$ and $\angle c$, are called **vertical angles.** $\angle b$ and $\angle d$ are also vertical angles.



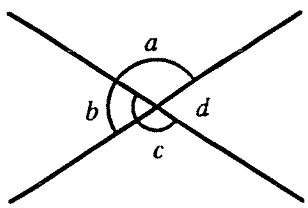
Problem 3 Find the measure of $\angle a$ and $\angle c$, if $\angle b = 40°$.

When two straight lines intersect as in the diagram to the right, the following generalizations hold true, regardless of the measure of $\angle b$.

$$\angle a = 180° - \angle b$$

$$\angle c = 180° - \angle b$$

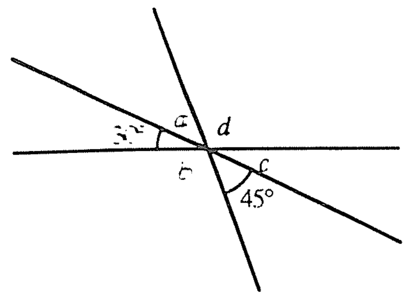
Therefore, $\angle a = \angle c$.



Problem 4 Following the procedure we adopted above, explain why $\angle b = \angle d$.

Property of Vertical Angles Vertical angles are equal.

Problem 5 In the diagram to the right three straight lines intersect. Find the measure of $\angle a$, $\angle b$, $\angle c$, $\angle d$.

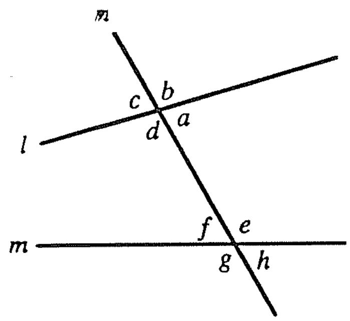


2 Angles and Parallel Lines

Corresponding Angles and Alternate Interior Angles

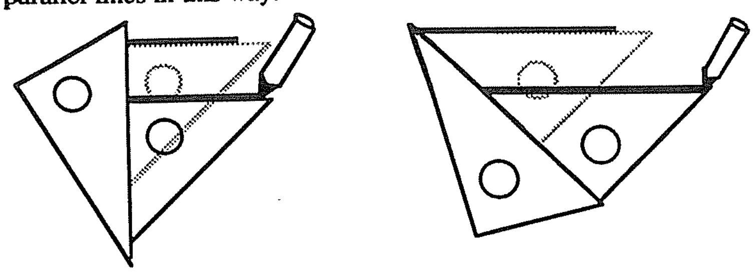
The diagram to the right shows a straight line n intersecting two straight lines l and m. $\angle$a and $\angle$h are said to be **corresponding angles.** Other pairs of corresponding angles are $\angle b$ and $\angle e$, $\angle c$ and $\angle f$, and $\angle d$ and $\angle g$

$\angle a$ and $\angle f$ are said to be **alternate interior angles.** $\angle d$ and $\angle e$ are also alternate interior angles.

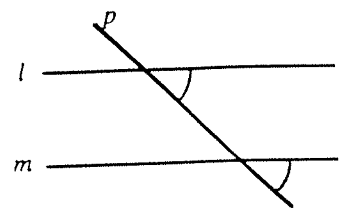


Parallel Lines and Corresponding Lines

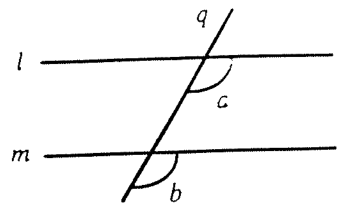
Problem 1 We can draw parallel lines by using drafting triangles. Draw some parallel lines in this way.



If we draw two straight lines l and m so that corresponding angles adjacent to them form another straight line p, as in the diagram to the right, then l and m are parallel.



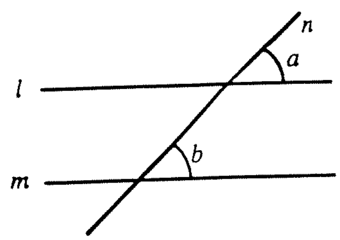
Moreover, if we draw a straight line q which crosses parallel lines l and m, the corresponding angles thus formed are equal. For example, $\angle a = \angle b$.



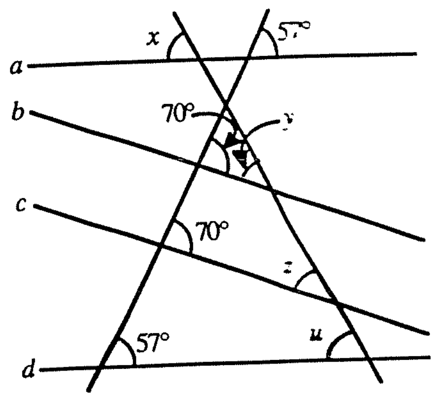
Thus, the following generalizations hold true:

(1) If $\angle a = \angle b$, then $l \parallel m$.

(2) If $l \parallel m$, then $\angle a = \angle b$.



Problem 2 Using the symbol $\parallel$, indicate which lines in the diagram at the right are parallel. Which of the angles $\angle x$, $\angle y$, $\angle z$, $\angle u$ are equal?

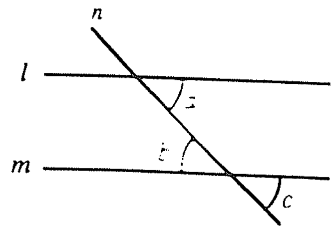


Parallel Lines and Alternate Interior Angles

When two straight lines are parallel, their alternate interior angles are equal. This can be explained by the following line of reasoning.

Let us assume that $l \parallel m$ in the diagram to the right. Because $\angle a$ and $\angle c$ are corresponding angles,

$$\angle a = \angle c$$



Because $\angle b$ and $\angle c$ are vertical angles,

$$\angle b = \angle c$$

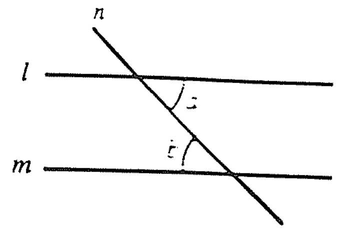
Hence, $\angle a = \angle b$

Problem 3 In the diagram to the right, if

$$\angle a = \angle b,$$

then

$$l \parallel m$$



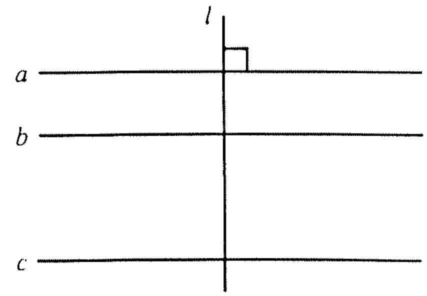
Explain this by reasoning from the fact that the corresponding angles of the two lines are equal.

Let us summarize what we have learned so far about the relations between angles and parallel lines.

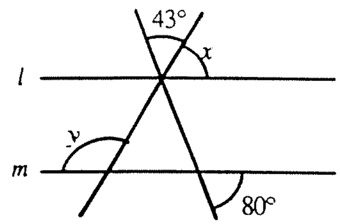
> **Properties of Parallel Lines**
>
> If a straight line intersects two parallel lines,
>
> (1) the corresponding angles are equal;
> (2) the alternate interior angles are equal.

> **Necessary Conditions for Parallel Lines**
>
> If a straight line intersects two other lines, those two lines are parallel if either of the following conditions holds true:
>
> (1) the corresponding angles of the two lines are equal;
> (2) the alternate interior angles of the two lines are equal.

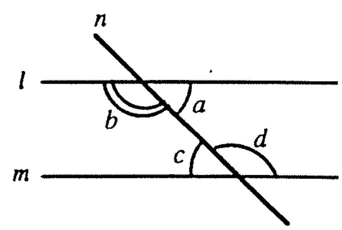
Problem 4 Let us assume that $a \parallel b$ and $a \parallel c$ in the diagram to the right. If we draw a straight line l perpendicular to a, then l is also perpendicular to b and c. Demonstrate that this is true using the properties of parallel lines. Then explain why $b \parallel c$.



Problem 5 Assuming that $l \parallel m$ in the diagram to the right, find the measure of $\angle x$, $\angle y$.



Problem 6 If $l \parallel m$ in the diagram to the right, then $\angle a + \angle d = 2\angle R$. Explain why this is so.

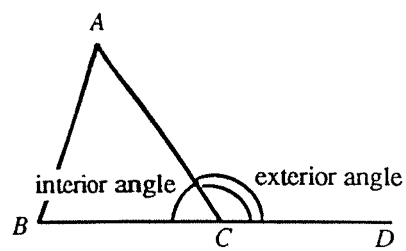


③ Interior and Exterior Angles of Triangles

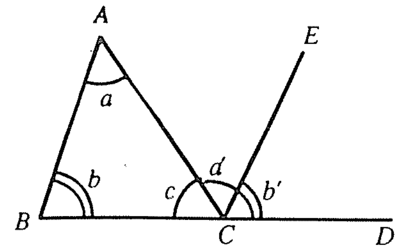
We use the symbol Δ to designate a triangle ABC as ΔABC.

In ΔABC in the diagram to the right, $\angle A$, $\angle B$, and $\angle C$ are referred to as the **interior angles** of ΔABC.

If we let CD be the extension of side BC, $\angle ACD$ is said to be an **exterior angle** of ΔABC at vertex C.



If we draw a straight line CE parallel to side AB through vertex C, as in the diagram to the right, then because alternate interior angles of parallel lines are equal,



$\angle a = \angle a'$

Because corresponding angles of parallel lines are equal,

$\angle b = \angle b'$

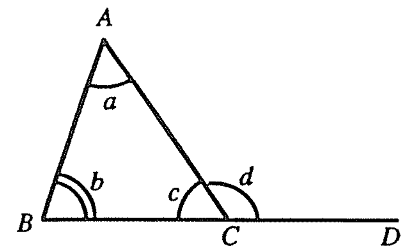
Because $\angle ACD = \angle a' + \angle b'$,

$$\angle ACD = \angle a + \angle b$$

Interior and Exterior Angles of Triangles

An exterior angle of a triangle is equal to the sum of the two interior angles which are not adjacent to it.

We can apply this generalization to the diagram to the right:

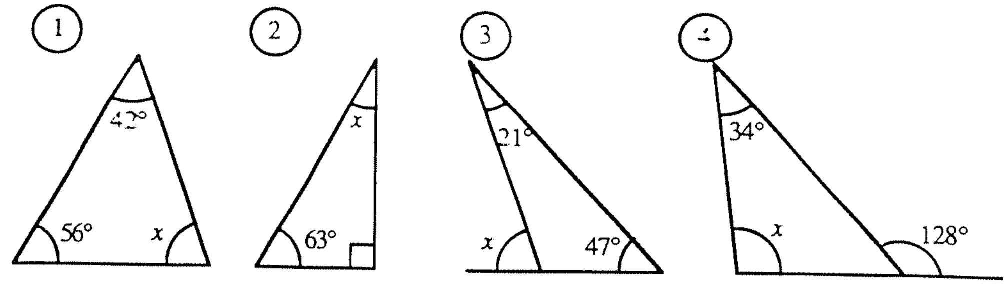


$$\angle a + \angle b + \angle c = \angle d + \angle c$$
$$= \angle BCD$$
$$= 2\angle R$$

The Sum of the Interior Angles of a Triangle

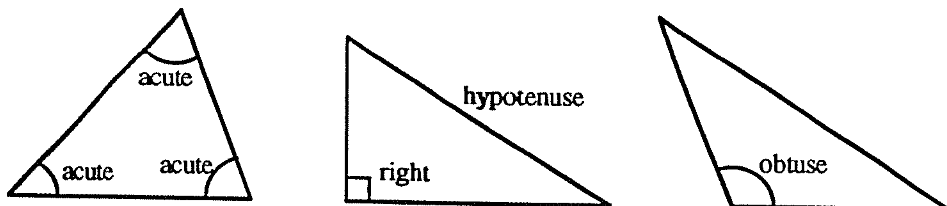
The sum of the three interior angles of a triangle is $2\angle R$.

Problem 1 Find the measure of $\angle x$ in the diagram below.



Problem 2 No triangle can have two interior angles such that each is either a right angle or obtuse angle. Explain why this is so.

We classify triangles according to the largest angle they contain. A triangle whose largest angle is acute is called an **acute triangle**; if the largest angle is a right angle, it is called a **right triangle**; if the largest angle is obtuse, it is called an **obtuse triangle**. The side of a right triangle opposite the right angle is called the **hypotenuse**.



acute triangle   right triangle   obtuse triangle

Problem 3 In a right triangle what is the sum of the interior angles which are not right angles?

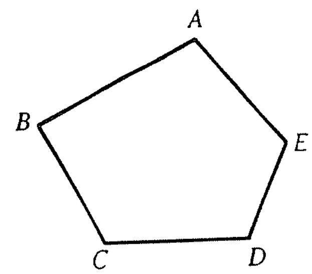
Problem 4 What kind of triangle – acute, right, or obtuse – can contain the following two interior angles?

(1) 30°, 80°   (2) 20°, 30°   (3) 50°, 40°

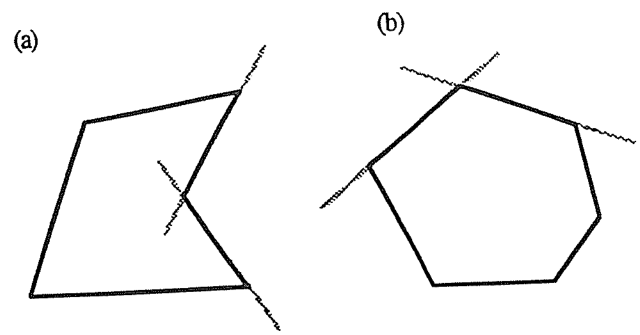
4 Interior and Exterior Angles of Polygons

Polygons

Plane figures made up of several line segments, such as triangles, quadrilaterals, and pentagons, are called polygons. We designate polygons by stringing together the names of their vertices, for example, pentagon *ABCDE*.

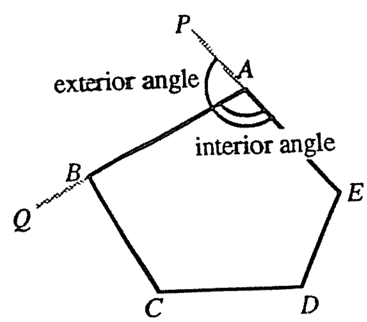


Note: In some polygons extensions of the sides are located in the interior of the polygon, as in diagram (a) below, and in other polygons the extensions are not located in the interior, as in diagram (b). From now on, when we speak of polygons, we will exclude figures like the one in diagram (a), unless otherwise specified.



The angle formed inside each vertex of a polygon by the two adjacent sides is called an interior angle of that polygon. $\angle BAE$ in the diagram to the right is an interior angle at vertex *A* of pentagon *ABCDE*.

The angle formed at each vertex of a polygon by one adjacent side and the extension of the other adjacent side is called an exterior angle at that vertex. $\angle BAP$ in the diagram to the right is an exterior angle at vertex *A* of pentagon *ABCDE*.

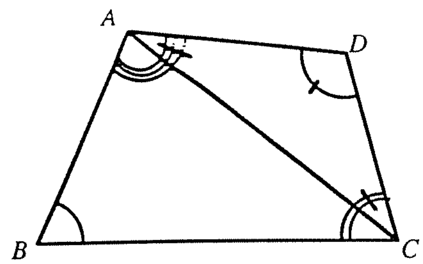


Problem 1 In the diagram above, if we extend BA and create another exterior angle at vertex A, that angle will be equal to $\angle BAP$. Explain why this is so.

Problem 2 In the diagram above, identify an exterior angle at vertex B.

The Sum of the Interior Angles of a Quadrilateral

Quadrilateral $ABCD$ can be divided into two triangles by drawing diagonal AC. If we add all the interior angles of these two triangles, we get the sum of the four interior angles of the original quadrilateral.



Since the sum of the interior angles of a triangle is $2\angle R$, the sum of the four interior angles of quadrilateral $ABCD$ is

$$2\angle R \times 2 = 4\angle R$$

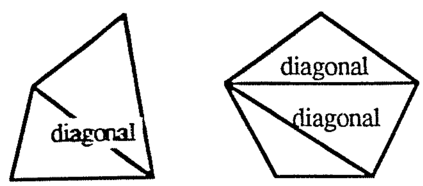
> **The Sum of the Interior Angles of a Quadrilateral**
>
> The sum of the four interior angles of a quadrilateral is $4\angle R$.

Problem 3 If three of the interior angles of a quadrilateral are right angles, the fourth angle must also be a right angle. Explain why this is so.

Polygons and Diagonals

Line segments which connect non-adjacent vertices of a polygon are called diagonals of that polygon.

The diagram to the right shows that quadrilaterals and pentagons can be divided into triangles by diagonals drawn from one vertex.



Thus, polygons can be divided into several triangles by drawing diagonals from one vertex.

Problem 4 Quadrilaterals, pentagons, hexagons, heptagons, and octagons can be divided into triangles by drawing diagonals from one vertex. Fill in the blanks in the following table.

	Quadrilateral	Pentagon	Hexagon	Heptagon	Octagon
Vertices	4	5			
Diagonals from one vertex	1	2			
Triangles	2	3			

The Sum of the Interior Angles of Polygons

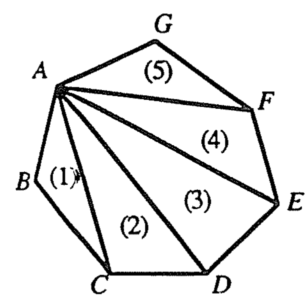
Problem 5 From the table above, what kind of relation can we establish between the number of vertices and the number of diagonals? Between the number of vertices and the number of triangles?

In general, the number of diagonals we can draw from one vertex is ([number of vertices] – 3), and the number of triangles that result is ([number of vertices] – 2). The sum of the angles of these triangles is the sum of the interior angles of the original polygon.

Because the sum of the interior angles of one triangle is $2\angle R$, the sum of the interior angles of any polygon can be found by means of the following formula:

$$2\angle R \times ([\text{number of vertices}] - 2)$$

The following generalization holds true for any polygon of n vertices, that is, an n-sided figure.



The Sum of the Interior Angles of an n-sided Figure

The sum of the interior angles of an n-sided figure is $2(n-2)\angle R$.

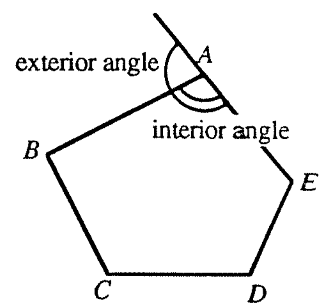
Problem 6 Find the sum of the interior angles of a hexagon, a heptagon, and an octagon.

The Sum of the Exterior Angles of Polygons

Let's consider the sum of the exterior angles of a pentagon.

Note: When we say the sum of the exterior angles, we mean the sum obtained by adding one exterior angle at each vertex.

Problem 7 In a pentagon, what is the sum of the exterior and interior angles at one vertex?



The sum of the exterior and interior angles at any vertex of a pentagon is $2\angle R$. Therefore, if we add up the sums of the interior and exterior angles at all five vertices, we get

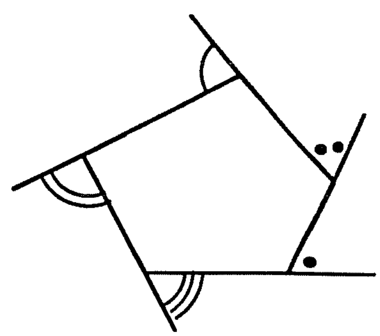
$$2\angle R \times 5 = 10\angle R \qquad (1)$$

The sum of only the five interior angles is

$$2(5-2)\angle R = 6\angle R \qquad (2)$$

We can then find the sum of the exterior angles of a pentagon by subtracting (2) from (1). Thus,

$$10\angle R - 6\angle R = 4\angle R$$

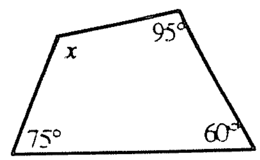


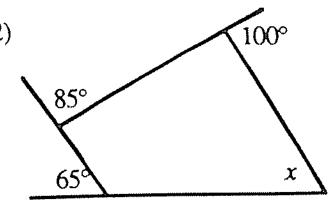
Problem 8 Find the sum of the exterior angles of a triangle and a hexagon as above.

The Sum of the Exterior Angles of a Polygon
The sum of the exterior angles of a polygon is $4\angle R$.

Problem 9 Find the measure of $\angle x$ in the diagrams below.

(1) (2)





Exercises

1. What is the measure of one exterior angle of a regular decagon? What is the measure of one interior angle?

2. How many sides does a regular polygon have if its interior angles are each 160°?

3. If $l \parallel m$ in the diagram to the right, find the measures of $\angle x$ and $\angle y$.