Beginning Algebra Continued

Second Edition

Raymond Louis Tenebruso

ABSTRACT. A good introduction to school algebra −one that is worth the student's time and effort− should leave the student believing that algebra is exactly as she knows it should be. The student's experience should naturally involve wonder and discovery while instilling the ideas, skills, understanding, and intellectual tendencies that will bring success in mathematics throughout high school and college. The author has taught school mathematics to grades 4 through various years of calculus. In doing so, he has seen children display intellectual sophistication that they are supposed to be too young to have. This text does not assume such sophistication, but provides opportunities for it to appear and be enjoyed. The author's deeply held belief that the student, although less experienced, is a peer in the quest for mathematical truth has produced a book respectful of the student's intellect, curiosity, wonder, and humanity. The text includes no fewer than 1400 problems. Answers to all problems and full solutions to some are provided in the Appendix. Most of these problems are practice. Some problems ask the student for a proof or a reasoned explanation.

BEGINNING ALGEBRA CONTINUED, Second Edition

ISBN: 978-0-578-16796-1

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Typeset using LaTeX

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Preface

An equation means nothing to me
unless it expresses a thought of God.
—Srinivasa Ramanujan

This book picks up where *Beginning Algebra* left off. Hence the catchy title. It does not include review of topics treated in *Beginning Algebra*. The full text of *Beginning Algebra* is accessible on the World Wide Web, so that a student may review topics without having to purchase the book.

The topics covered in this book and the manner in which they are treated can leave you prepared for success in future mathematics courses. Granted, *will* instead of *can* in the preceding sentence would have provided a welcome sense of security. But it would have been deceptive. The essential ingredient that turns "*can*" into "*will*" is your participation. Here are a few suggestions.

(1) Begin assigned exercise problems only after you have studied the material in the book preceding the exercise.

(2) Study by working along with the book on a piece of scrap paper.

(3) Take advantage of the discussion in the book. The story of mathematics unfolds in a very reasonable way. The better you understand the story, the less you will rely on memory.

(4) Work examples along with the book. They give you experience applying ideas and sometimes a chance to explore those ideas.

(5) Copy the problem onto your own paper. You will become fluent in the language of mathematics sooner.

(6) The answer to every problem in the book is in the back of the book. Check your answer to each problem as soon as you finish each problem. If you have a misunderstanding, you will catch it right away. Waiting until you finish all the problems before you check your answers could amount to your having spent considerable effort solidifying a misunderstanding.

(7) View exercise problems a tool for learning, not as a kind of self test.

(8) Do assignments in time for the next class. Then you will be prepared to understand new material.

(9) Join the class discussion. Ask your teacher and classmates questions. Try to respond to their questions and assertions.

The book *Beginning Algebra* is accessible on line. If you wish to review a topic from *Beginning Algebra*, visit http://www.mnrt.net/Publications/ .

I wish to thank the 2014-2015 8^{th} grade class at *Madison Country Day School* for their valuable insights and their patience during the first use of this book.

I thank George Ekman who quite thoroughly checked calculations in the text and answers in the appendix. Errors that remain in this second edition are due no doubt to my failure to correct all that were found.

Ray Tenebruso

Chapter 1

Integers

As you have seen, the integers provide the foundation for the other important kinds of numbers, such as the rational numbers. The study of the integers is an area of mathematics called "Number Theory". Although the integers might seem simple compared to other kinds of numbers, they have furnished some of the most compelling problems in mathematics as well as some of the most difficult. The ideas and techniques that you will learn in this chapter will be valuable to you throughout your study of mathematics.

1.1. Factors, divisors, and multiples

From previous grades, you are already familiar with the ideas of factor and multiple. Even so, it will not hurt to be sure we are all talking about the same ideas. If a number is a product of several positive integers, we call those several positive integers "factors" of the number. The number is a "multiple" of each of its factors.

Example 1.1

Find all the factors of 12.

Solution

Since $3 \cdot 4 = 12$, 3 and 4 are each factors of 12. Of course, 12 also equals $2 \cdot 6$, so 2 and 6 are also factors of 12. Let us not overlook $1 \cdot 12 = 12$, so that 1 and 12 get to be factors of 12, too. Unless we have missed some, the factors of 12 are $1, 2, 3, 4, 6, 12$. ■

We also know that:

- 12 is a multiple of 1, because $12 = 1 \cdot 12$,
- 12 is a multiple of 2, because $12 = 2 \cdot 6$,
- 12 is a multiple of 3, because $12 = 3 \cdot 4$,
- 12 is a multiple of 4, because $12 = 3 \cdot 4$,
- 12 is a multiple of 6, because $12 = 2 \cdot 6$,
- 12 is a multiple of 12, because $12 = 1 \cdot 12$.

Example 1.2

Find all the multiples of 2.

Solution

All the multiples are 2 are provided by $2n$ where n is a positive integer. As n ranges over the positive integers

$$1, 2, 3, \cdots,$$

$2n$ takes the values

$$2, 4, 6, \cdots.$$

Example 1.3

Find the first 6 multiples of 3.

Solution

The first 6 multiples of 3 are provided by $3n$ where $n = 1, 2, 3, 4, 5, 6$. They are $3, 6, 9, 12, 15, 18$. ■

When a number can be divided without remainder by a positive integer that we will call p, we say that p is a "divisor" of the number.

Example 1.4

6 is a divisor of 72, because $72 \div 6 = 12$ with no remainder. But, 6 is not a divisor of 75 because $75 \div 6$ leaves a remainder of 3. ■

We may as well state as definitions these ideas about what are factors, divisors, and multiples.

Definition 1.1 (Factor)
A positive integer p is a **factor** of a number N, if $N = p \cdot q$, where q is also a positive integer.

Definition 1.2 (Divisor)
A positive integer p is a **divisor** of a number N, if $N \div p$ has remainder 0. When p is a divisor of N, we say "p divides N" or "N is divisible by p".

Definition 1.3 (Multiple)
A number N is a **multiple** of a positive integer p, if $N = p \cdot q$, where q is also a positive integer.

Exercise 1.1

1. List all of the factors of each of the following numbers.

a) 6	c) 12	e) 36
b) 21	d) 45	f) 7

2. List the first 6 multiples of each of the following numbers.

a) 5	c) 1	e) 7
b) 6	d) 10	f) 101

3. Is 7 a divisor of 42?
4. Is 8 a divisor of 63?
5. What is the largest number less than 1000 that is divisible by 6?
6. Prove that 7 is not a divisor of 23.
7. Of the multiples of 3 less than 30, how many are also multiples of 6?
8. What is the smallest number that is a multiple of 3 and 5?
9. What is the smallest number that is a multiple of 3 and 9?
10. What number is the greatest factor of both 12 and 28?
11. In Example 1.2 on the facing page, we found all the multiples of 2. Write all the odd numbers.

1.2. Prime numbers

Prime numbers have fascinated human beings since our ancestors first noticed them. Prime numbers occur in no recognizable pattern. As hard as people have tried, no one has discovered a formula that produces all and only prime numbers. A theorem called the Fundamental Theorem of Arithmetic guarantees that every integer is either a prime number or a product of prime numbers. This means the prime numbers are the building blocks of the integers. In spite of this, it is very difficult to discover the prime factors of a very large non-prime number. This fact has a practical application. The cryptographic scheme that protects millions of electronic communications every day succeeds because of the extreme difficulty of finding the prime factors of very large numbers.

Definition 1.4 (Prime number)
A number is a **prime number** if it has exactly two factors, 1 and itself.

Definition 1.5 (Composite number, non-prime number)
A number that is not a prime number is called a **non-prime number** or a **composite number**.

The next theorem, which will not be proved here, is called the Fundamental Theorem of Arithmetic. The name suggests the importance of its role.

Theorem 1.1 (Fundamental Theorem of Arithmetic)
Every positive integer (except the number 1) can be represented in exactly one, way apart from rearrangement, as a product of one or more primes.

1.2.1. Sieve of Eratosthenes

One method of finding the prime numbers up to a number N is called the Sieve of Eratosthenes. It works like this. List the numbers from 2 to N. Leave 2 alone, but cross out all other multiples of 2. The first number after 2 that is not crossed out must be a prime number; indeed, it is the number 3. Leave 3 alone, but cross out all other multiples of 3. The first number after 3 that is not crossed out must be a prime number. In fact it is 5. Leave 5 alone, but cross out all other multiples of 5. Continue this process until every number up to N has been either left alone or crossed out. The numbers that are not crossed out are the prime numbers from 2 to N. The method of the Sieve of

Eratosthenes for the first 100 positive integers is illustrated in Appendix A on page 227.

If you use the Sieve of Eratosthenes to find all the prime numbers less than 100, you will find they are

$$2,3,5,7,11,13,17,19,23,29,31,37,41,$$
$$43,47,53,59,61,67,71,73,79,83,89,97.$$

You should know these numbers by heart. They will become familiar friends.

Example 1.5

Is 27 a prime number?

Solution

No. 3 is a factor of 27, so 27 has a factor other than 1 and itself.

Example 1.6

Is 23 a prime number?

Solution

Yes. The only factors of 23 are 1 and 23.

1.3. Prime factorization

The Fundamental Theorem of Arithmetic guarantees that every positive integer is either a prime number or a product of prime numbers. Examples 1.7 to 1.10 on pages 5–7 demonstrate a systematic procedure to find a number's prime factors.

Example 1.7

Find the prime factorization of 12.

Solution

The first prime number is 2. Keep dividing by 2 until a number not divisible by 2 is obtained. That number is 3. The next prime number after 2 is 3. Divide by 3. The appearance of the number 1 terminates the process.

$$\begin{array}{r|l} 2 & 12 \\ 2 & 6 \\ 3 & 3 \\ & 1 \end{array}$$

Therefore, $12 = 2 \cdot 2 \cdot 3$.

Example 1.8

Find the prime factorization of 50.

Solution

The first prime number is 2. Division by 2 produces 25 which is not divisible by 2. The next prime number after 2 is 3. But, 25 is not divisible by 3. The next prime number after 3 is 5. Keep dividing by 5. The appearance of the number 1 terminates the process.

$$\begin{array}{r|l} 2 & 50 \\ 5 & 25 \\ 5 & 5 \\ & 1 \end{array}$$

Therefore, $50 = 2 \cdot 5 \cdot 5$.

Example 1.9

Find the prime factorization of 455.

Solution

The first prime number is 2. But, 2 is not a divisor of 455. The next prime number is 3. But, again, no divisor. The next prime number is 5 and it does divide 455.

$$\begin{array}{r|l} 5 & 455 \\ 7 & 91 \\ 13 & 13 \\ & 1 \end{array}$$

Therefore, $455 = 5 \cdot 7 \cdot 13$.

Example 1.10

Completely factor 72.

Solution

$$\begin{array}{l} 2\,\underline{\lfloor 72} \\ \quad 2\,\underline{\lfloor 36} \\ \qquad 2\,\underline{\lfloor 18} \\ \qquad\quad 3\,\underline{\lfloor 9\ } \\ \qquad\qquad 3\,\underline{\lfloor 3\ } \\ \qquad\qquad\quad 1 \end{array}$$

Therefore, $72 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$.

The result $72 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$ can be written more compactly using exponents. We write three factors of 2 as 2^3. We write two factors of 3 as 3^2. So $72 = 2^3 \cdot 3^2$.

Example 1.11

Write 504 as a product of prime factors.

Solution

$$\begin{array}{l} 2\,\underline{\lfloor 504} \\ \quad 2\,\underline{\lfloor 252} \\ \qquad 2\,\underline{\lfloor 126} \\ \qquad\quad 3\,\underline{\lfloor 63} \\ \qquad\qquad 3\,\underline{\lfloor 21} \\ \qquad\qquad\quad 7\,\underline{\lfloor 7} \\ \qquad\qquad\qquad 1 \end{array}$$

Therefore, $504 = 2^3 \cdot 3^2 \cdot 7$.

Exercise 1.2

1. Write each of the following using exponents.

   a) $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$

   b) $2 \cdot 2 \cdot 2 \cdot 5 \cdot 5$

   c) $7 \cdot 11 \cdot 13 \cdot 13 \cdot 19$

   d) $2 \cdot 3 \cdot 7 \cdot 7 \cdot 7 \cdot 7$

   e) $2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 5 \cdot 11 \cdot 11$

2. Write each of the following numbers as a product of prime factors.

   a) 330

   b) 28

   c) 88

   d) 35

   e) 325

   f) 90

   g) 4125

   h) 1728

   i) 2646

   j) 132

3. Find each product.

   a) $2^3 5$

   b) 3^3

   c) $2 \cdot 3^3$

   d) $2^2 \cdot 3^2$

   e) 2^4

   f) $2^2 \cdot 5^2$

   g) 10^2

   h) 10^3

   i) 10^4

   j) 5^3

   k) $2 \cdot 3 \cdot 10^2$

   l) $2^2 \cdot 3 \cdot 7^2$

1.4. Greatest common divisor

The number which is the greatest common divisor of several numbers, must meet two conditions.

(1) The number must be a divisor of each of the several numbers.
(2) The number must be the greatest of those divisors.

The phrase "greatest common divisor" is abbreviated "GCD". The phrase "greatest common factor" is synonymous to "greatest common divisor" . The abbreviation for "greatest common factor" is "GCF".

Example 1.12

Find the greatest common divisor of 12 and 30.

Solution

The divisors of 12 are $1,2,3,4,6,12$.
The divisors of 30 are $1,2,3,5,6,10,15,30$.
The common divisors of 12 and 30 are $1,2,3,6$. Of these common divisors, 6 is the greatest.
Therefore, the greatest common divisor of 12 and 30 is 6. A compact way to write this is $GCD[12,30] = 6$.

Example 1.13

Find the greatest common divisor of 42 and 105.

Solution

The divisors of 42 are $1,2,3,6,7,14,21,42$.
The divisors of 105 are $1,3,5,7,15,21,35,105$.
The common divisors of 42 and 105 are $3,7,21$. Of these common divisors, 21 is the greatest.
Therefore, $GCD[42,105] = 21$.

When the two numbers whose GCD is desired do not have many divisors, as in examples 1.12 and 1.13, listing the divisors and choosing the greatest of the common divisors is a practical strategy. But, what about when the number of divisors is large? Or when the GCD of many numbers is desired?

For example, suppose we wish to know $GCD[420,660]$? Just finding all the divisors or 420 and of 660 is annoying. If you are not sure how annoying, go ahead and do it.

Do not despair. A little thought can eliminate a lot of computation. Example 1.14 shows a less laborious approach.

Example 1.14

Find the greatest common divisor of 420 and 660.

Solution

The prime factorizations of these numbers are not hard to obtain.

$$420 = 2^2 \cdot 3 \cdot 5 \cdot 7.$$
$$660 = 2^2 \cdot 3 \cdot 5 \cdot 11.$$

Each number has two factors of 2, one factor of 3 and one factor of 5. So, the greatest common divisor is $4 \cdot 3 \cdot 5 = 60$.

Example 1.15

Find the greatest common divisor of 120 and 140.

Solution

The prime factorizations of these numbers are:

$$120 = 2^3 \cdot 3 \cdot 5$$
$$140 = 2^2 \cdot 5 \cdot 7.$$

Each number has two factors of 2 and one of 5. So $GCD[120,140] = 20$. ■

Notice that 120 has 3 factors of 2, but 140 has only 2 factors of 2. So two, not three, factors of 2 appear in the greatest *common* factor of 120 and 140.

Suppose we have been provided the prime factorizations of two numbers, call them A and B, shown below. How do we determine which factors are present in the greatest common divisor, $GCD[A,B]$?

Although we usually do not write an exponent of 1, we will now, because it helps make the method obvious.

$$A = 2^3 \cdot 3^5 \cdot 5^1 \cdot 7^2 \cdot 13^4$$
$$B = 2^3 \cdot 3^3 \cdot 5^6 \cdot 7^3 \cdot 13^3$$

Remember the exponent tells how many of a factor there are. All the factors common to both A and B are $2^3, 3^3, 5^1, 7^2, 13^3$. So

$$GCD[A,B] = 2^3 \cdot 3^3 \cdot 5^1 \cdot 7^2 \cdot 13^3.$$

Example 1.16

Find the greatest common divisor of

$$A = 2^2 \cdot 3^4 \cdot 7^2 \cdot 11$$
$$B = 2^3 \cdot 3^3 \cdot 5^6 \cdot 7^3 \cdot 13.$$

Solution

Number A has no factor of 5 and no factor of 13, so 5 and 13 will not appear in $GCD[A,B]$. Since B has no factor of 11, the number 11 is not a common factor. So,

$$GCD[A,B] = 2^2 \cdot 3^3 \cdot 7^2.$$

■

If we define a number with an exponent of 0, we can write a simple rule for finding the greatest divisor.

Definition 1.6 (Zero exponent)
For any number x, $x^0 = 1$.

This definition will be fully discussed later on page 80.

Rule: to find the greatest common divisor of several numbers, write the prime factorization of each using the exponent 0 for missing factors. Then choose each factor using the smallest exponent.

Example 1.17

Find the greatest common divisor of 540 and 350.

Solution

$$540 = 2^2 \cdot 3^3 \cdot 5^1 \cdot 7^0$$

$$350 = 2^1 \cdot 3^0 \cdot 5^2 \cdot 7^1.$$

$$\therefore \quad GCD[540, 350] = 2^1 \cdot 3^0 \cdot 5^1 \cdot 7^0 = 2 \cdot 1 \cdot 5 \cdot 1 = 10.$$

Example 1.18

Find the greatest common divisor of $63, 84$, and 490.

Solution

$$63 = 2^0 \cdot 3^2 \cdot 5^0 \cdot 7^1$$
$$84 = 2^2 \cdot 3^1 \cdot 5^0 \cdot 7^1$$
$$490 = 2^1 \cdot 3^0 \cdot 5^1 \cdot 7^2.$$

$$\therefore \quad GCD[63, 84, 490] = 2^0 \cdot 3^0 \cdot 5^0 \cdot 7^1 = 7.$$

Example 1.19

Find the greatest common divisor of 14 and 15.

Solution

$$14 = 2^1 \cdot 3^0 \cdot 5^0 \cdot 7^1$$
$$15 = 2^0 \cdot 3^1 \cdot 5^1 \cdot 7^0.$$

$$\therefore \quad GCD[540, 350] = 2^0 \cdot 3^0 \cdot 5^0 \cdot 7^0 = 1.$$

Well, you probably already knew $GCD[14, 15] = 1$ before the work of Example 1.19. Numbers whose greatest common divisor is 1 are called “relatively prime” numbers.

Definition 1.7 (Relatively prime)
Numbers whose greatest common divisor is 1 are called **relatively prime** numbers.

Exercise 1.3

1. Find the greatest common divisor of each pair of numbers.

   a) $96, 80$
   b) $48, 72$
   c) $28, 98$
   d) $76, 57$
   e) $84, 63$
   f) $39, 52$
   g) $70, 28$
   h) $42, 78$
   i) $56, 70$
   j) $48, 32$
   k) $84, 78$
   l) $96, 72$

2. Find the greatest common divisor of each pair of numbers.

   a) $84, 126$
   b) $105, 70$
   c) $128, 160$
   d) $69, 34$
   e) $192, 96$
   f) $144, 156$
   g) $105, 168$
   h) $178, 80$
   i) $58, 116$
   j) $114, 152$
   k) $55, 143$
   l) $66, 198$

3. Find the greatest common divisor of set of numbers.

   a) $21, 92, 65$
   b) $78, 72, 66$
   c) $54, 72, 90$
   d) $96, 72, 60$
   e) $56, 98, 42$
   f) $51, 68, 85$
   g) $32, 64, 48$
   h) $55, 66, 77$
   i) $56, 40, 80$
   j) $69, 92, 46$
   k) $34, 85, 51$
   l) $48, 80, 64$

4. The GCD of a pair of numbers is 4 and the sum of the pair of numbers is 32. Find all pairs.

5. The GCD of a pair of numbers is 18 and the sum of the pair of numbers is 324. Find all pairs.

1.5. Least common multiple

You have been finding the least common multiple of sets of numbers for several years. Every time you wished to add factions whose denominators were not identical, you found a least common denominator. That denominator was the least common multiple of all the denominators. We denote the least common multiple of numbers A and B by writing "$LCM[A,B]$".

Often it is practical, as in Example 1.20, to simply write or imagine the multiples of two numbers in numeric order from least to greatest. The first multiple that is common is the least common multiple.

Example 1.20

Find the least common multiple of 6 and 15.

Solution

The first few multiples, in order, of 6 and 15 are

$$6, 12, 18, 24, 30, 36 \cdots$$
$$15, 30, 45 \cdots .$$

The first common multiple we meet is 30 and this is the least common multiple. $LCM[6,15] = 30$. ■

Not being content to leave well enough alone, we wonder what is $LCM[36,196]$. We list the multiples of each number.

For 36:
$36, 72, 108, 144, 180, 216, 252, 288, 324, 360, 396, 432, 468, 504, 540, 576, 612, 648, 684, 720, 756, 792, 828, 864, 900, 936, 972, 1008, 1044, 1080, 1116, 1152, 1188, 1224, 1260, 1296, 1332, 1368, 1404, 1440, 1476, 1512, 1548, 1584, 1620, 656, 1692, 1728, 1764, \cdots$

For 196:
$196, 392, 588, 784, 980, 1176, 1372, 1568, 1764, \cdots$

Is there less exhausting way to obtain $LCM[36,196] = 1764$?

Often, we get insight into a number by showing its building blocks. That is, by writing the prime factorization of the number. Doing so for 36 and 196,

$$36 = 2^2 \cdot 3^2$$
$$196 = 2^2 \cdot 7^2.$$

A multiple of a number must have all the factors of that number. Multiples of 36 must have at least two factors of 2 and at least two factors of 3. Every multiple of 196 must have at least two factors of 2 and at least two factors of 7. The number whose factors are

$$2^2 \cdot 3^2 \cdot 7^2$$

has just the needed factors and no more. It is the least common multiple of 36 and 196. Therefore $LCM[36, 196] = 2^2 \cdot 3^2 \cdot 7^2 = 4 \cdot 9 \cdot 49 = 1764$.

Example 1.21

Find the least common multiple of 363 and 99.

Solution

$$363 = 3^1 \cdot 11^2$$
$$99 = 3^2 \cdot 11^1.$$

Therefore, $LCM[363, 99] = 3^2 \cdot 11^2 = 9 \cdot 121 = 1089$.

Example 1.22

Find $LCM[140, 165]$.

Solution

$$140 = 2^2 \cdot 3^0 \cdot 5^1 \cdot 7^1 \cdot 11^0$$
$$165 = 2^0 \cdot 3^1 \cdot 5^1 \cdot 7^0 \cdot 11^1.$$

Therefore, $LCM[140, 165] = 2^2 \cdot 3^1 \cdot 5^1 \cdot 7^1 \cdot 11^1$. ■

Rule for finding the least common multiple of several numbers. Write the prime factorizations of each number. Choose every number that appears in either factorization and use the greatest exponent taken by the number.

Exercise 1.4

1. Find the least common multiple of each pair of numbers.

   a) $36, 45$
   b) $18, 48$
   c) $36, 24$
   d) $12, 42$
   e) $24, 40$
   f) $14, 21$
   g) $40, 30$
   h) $20, 22$
   i) $21, 28$
   j) $45, 10$
   k) $12, 16$
   l) $48, 36$

2. Find the least common multiple of each set of numbers.

   a) $30, 50, 20$
   b) $35, 10, 45$
   c) $48, 24, 36$
   d) $22, 44, 33$
   e) $22, 6, 18$
   f) $12, 20, 16$

3. The LCM of 30 and another number is 840. Find the other number.

4. The GCD of a pair of numbers is 2 and the LCM of the pair is 90. Find all pairs.

5. Suppose that a cat returns to the same place in a barn every 21 days and that a mouse returns to that spot every 6 days. If the cat just met the mouse, but failed to catch it, how many days later will the cat get her next chance to catch the mouse?

1.6. Building integers

In Section 1.2 on page 4 we said that the prime numbers are the building blocks of the integers. We considered how to decompose a composite number into its prime factors. Now we consider building up instead of tearing down. There is an obvious but helpful idea. If all the prime factors of an integer N are included in $\{p_1, p_2, p_3, \cdots, p_n\}$, then the prime factors of a factor of N must belong to the set $\{p_1, p_2, p_3, \cdots, p_n\}$.

For example, the prime factors of 30 are $2, 3, 5$. If n is a divisor of 30, then n can have no more than one factor of 2, one factor of 3, and one factor of 5.

1.6.1. Counting factors

If we know the prime factors of a number, we can predict how many divisors the number has. The following examples show how.

Example 1.23

How many factors (prime and non-prime) has the number 30? (The prime factors of 30 are $2, 3, 5$.)

Solution

Our project is to build a factor of 30. We do this as a sequence of tasks.

Task 1: Choose some number of 2s. The are two ways to do this. Choose no 2, $2^0 = 1$ or choose one 2, $2^1 = 2$. So, . . . 2 ways.

Task 2: Choose some number of 3s. The are two ways to do this. Choose no 3, $3^0 = 1$ or choose one 3, $3^1 = 3$. So, . . . 2 ways.

Task 3: Choose some number of 5s. The are two ways to do this. Choose no 5, $5^0 = 1$ or choose one 5, $5^1 = 5$. So, . . . 2 ways.

There are $2 \cdot 2 \cdot 2 = 6$ ways to complete the project. Therefore, there are 6 factors of 30.

Example 1.24

How many divisors has the number 504?

Solution

Note that $504 = 2^3 \cdot 3^2 \cdot 7^1$.

Project: Build a divisor of 504.

Task 1: Choose $0, 1, 2$, or 3 factors of 2. . . . 4 ways.
Task 2: Choose $0, 1$, or 2 factors of 3. . . . 3 ways.
Task 3: Choose 0 or 1 factor of 7. . . . 2 ways.

There are $4 \cdot 3 \cdot 2 = 24$ ways to complete the project. Therefore, there are 24 factors of 504.

1.6.2. Building divisors

Each divisor of a number must be a product of some combination of prime factors of that number. For example, 6 is a divisor of 12. The divisor 6 is built using one factor of 2 and one factor of 3. The divisor 1 is built using no factor of 2, $2^0 = 1$, and no factor of 3, $3^0 = 1$.

Example 1.25

Find all the divisors of 18.

Solution

The prime factorization of 18 is $2 \cdot 3^2$. We expect 6 divisors. Writing every combination of factors of 12,

$$2^0 \cdot 3^0 = 1 \qquad 2^1 \cdot 3^0 = 2$$
$$2^0 \cdot 3^1 = 3 \qquad 2^1 \cdot 3^1 = 6$$
$$2^0 \cdot 3^2 = 9 \qquad 2^1 \cdot 3^2 = 18.$$

Example 1.26

Find all the factors of 100.

Solution

The prime factorization of 100 is $2^2 \cdot 5^2$. We expect 9 divisors. Writing every combination of factors of 100,

$2^0 \cdot 5^0 = 1$	$2^1 \cdot 5^0 = 2$	$2^2 \cdot 5^0 = 4$
$2^0 \cdot 5^1 = 5$	$2^1 \cdot 5^1 = 10$	$2^2 \cdot 5^1 = 20$
$2^0 \cdot 5^2 = 25$	$2^1 \cdot 5^2 = 50$	$2^2 \cdot 5^2 = 100$

Example 1.27

Find all the factors of 80.

Solution

The prime factorization of 40 is $2^4 \cdot 5^1$. We expect 10 divisors.

$2^0 \cdot 5^0 = 1$	$2^1 \cdot 5^0 = 2$	$2^2 \cdot 5^0 = 4$	$2^3 \cdot 5^0 = 8$	$2^4 \cdot 5^0 = 16$
$2^0 \cdot 5^1 = 5$	$2^1 \cdot 5^1 = 10$	$2^2 \cdot 5^1 = 20$	$2^3 \cdot 5^1 = 40$	$2^4 \cdot 5^1 = 80$

Exercise 1.5

How many divisors has each number

1. 72
2. 40
3. 126
4. 560

Write all divisors of the number

5. 8
6. 98
7. 54
8. 675

Chapter 2

The straight line

2.1. Real numbers

Our intuition of a line, straight or otherwise, is that it is a continuous kind of thing. When we draw a line, we intend the drawing to have no skips or gaps. If you look at the drawn line under great enough magnification, you will probably observe many skips and gaps. That is the nature of pens, pencils, and paper. But, we do not really care about that, because the drawn line is just a picture to remind us of *the line we have in mind*. That line is free of skips or gaps. It is perfect continuity.

We wish to develop the connection of algebra and geometry. There is a catch, though. We have officially discovered so far in *Beginning Algebra* the rational numbers. But there are more points on the line than there are rational numbers. Even for a very short piece of a line. Imagine the portion of the number line from 1 to 2. There are not enough rational numbers to label each point on that little piece. In other words, there are points on the line for which no rational number exists. We prove that in Chapter 3.

There is a larger set of numbers called the *real numbers* that does contain a number for every point on the line. The real numbers are in a 1-1 correspondence with the points of a line. These are the numbers we need. Right now. The trouble is, there is no way that the real numbers can be adequately defined until you reach a certain level of mathematics. To get to that level, you have to learn and use mathematics that requires the real numbers. How's that for a bind?

Here is what we will do. We will use the name "real numbers", knowing that there is a 1-1 correspondence of the real numbers and the points on a line. The set of real numbers includes all the integers and all the rational numbers. It includes every number you meet in this book. Every name –except the one described in the next paragraph– that you have seen means a real number. For example, $3, \frac{2}{5}, -7, \sqrt{2}, 0, \sqrt[3]{7}$, and π all name real numbers.

Who is not a real number? If you see a name that includes the symbol "i", that name might (you will know by the context) refer to a kind of number that is called a "complex" or "imaginary" number. It is not in the set of real numbers. Names of complex numbers look like this "$2+9i$" or "$4i$". Other than this paragraph, complex numbers do not appear in this book.

From this page on, the word "number" with no qualification such as "rational" refers to the real numbers. We will discuss irrational numbers in Chapter 3. Figure 2.1 shows the hierarchy of the numbers. A set of numbers is contained as a subset in the set above it.

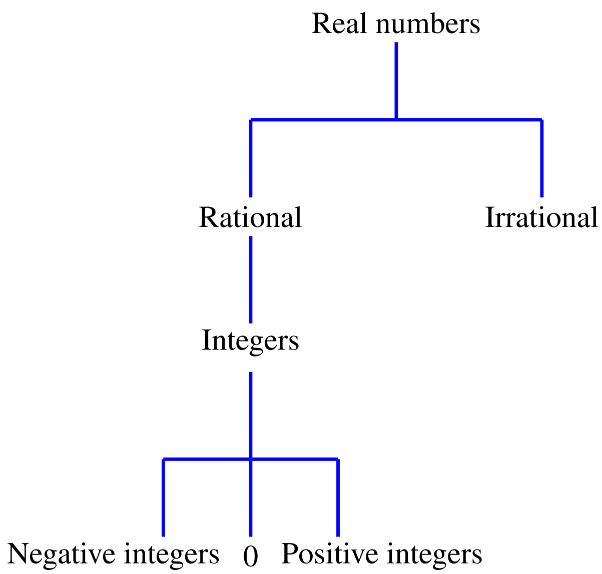


FIGURE 2.1. Hierarchy of numbers

2.2. What counts as straight?

Some lines are straight. But, there are all manner of lines that are not straight. For example, the graph of $y = x^5 - 2x^3$ shown in Figure 2.2 on the following page is a line, but not straight.

Figure 2.3 on the next page shows the accurately drawn graphs of two functions. One graph is a straight line and the other is not. Can you tell the difference? Figure 2.4 provides a closer look at a small, $0 \leq x \leq 0.4$, portion of the graphs of Figure 2.3 magnified 10 times. Can you see the difference, now?

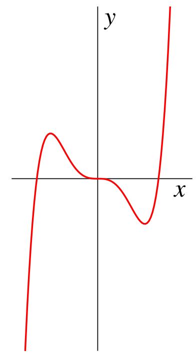
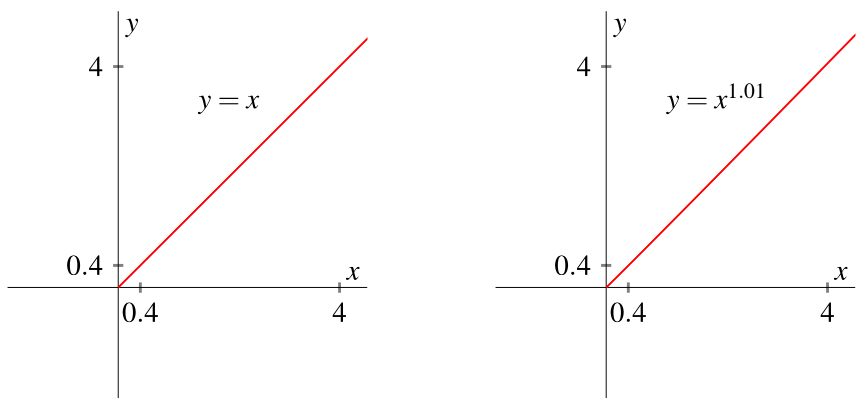
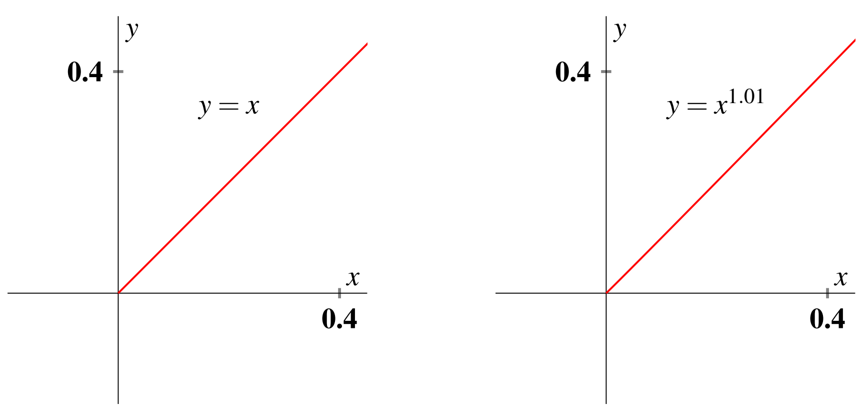


FIGURE 2.2. The graph of $y = x^5 - 2x^3$.



(a) $y = x$. Straight line. (b) $y = x^{1.01}$. No straight line.

FIGURE 2.3. Both lines appear straight. But one is not.



(a) $y = x$. Straight line. (b) $y = x^{1.01}$. No straight line.

FIGURE 2.4. Small portion of graphs of Figure 2.3 magnified 10 times.

Appearances can be deceiving. Both lines shown in Figure 2.4 look straight. The graphs provide no clue that one line is straight, the other not. Your mind and a little algebra can make clear that which your eyes cannot.

In the author's experience, student discussions of "What counts as straight?" nearly always reach the conclusion that a line whose direction never changes is a straight line.

The question then becomes "How to know whether or not a line changes direction?" This question has a precise answer: If the slope of the line is constant, the direction of the line is unchanging.

Definition 2.1 (Straight line)
A **straight line** is a line whose slope is constant. ■

Using Definition 2.1, we will prove Theorem 2.1,

Theorem 2.1
A non-vertical line in the xy-coordinate plane is a straight line if and only if it is the graph of $y = mx + b$, where m and b are constants and m is the slope of the line.

Proof. Let b be a constant. We must prove two statements.

(1) If the equation is $y = mx + b$, m a constant, then the line is straight.
(2) If the line is straight, then it's equation is $y = mx + b$, m a constant.

Part 1. Suppose the equation of a line is $y = mx + b$ where m is a constant. Let $P(x_1, y_1), Q(x_2, y_2)$ be any two points on the graph of $y = mx + b$ such that $x_1 \neq x_2$. See (a) Figure 2.5 on the following page. Then,

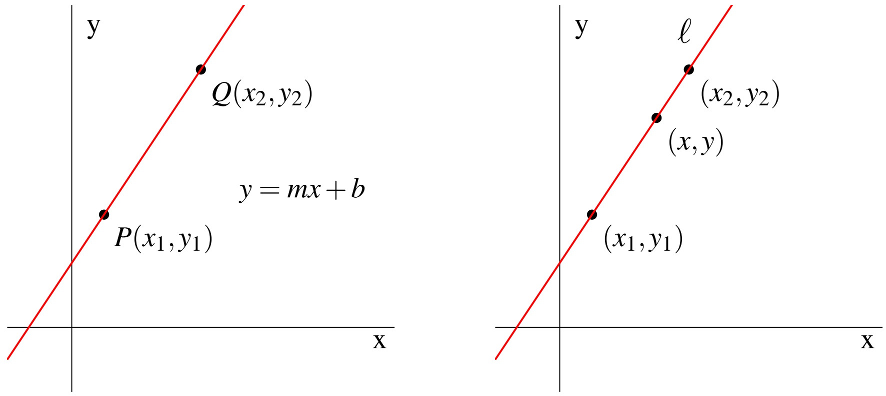
$$\text{slope using points } P \text{ and } Q = \frac{y_2 - y_1}{x_2 - x_1}.$$

Since, $y_1 = mx_1 + b$ and $y_2 = mx_2 + b$, we substitute

$$\begin{aligned}\frac{(mx_2 + b) - (mx_1 + b)}{x_2 - x_1} &= \frac{m(x_2 - x_1)}{x_2 - x_1} \\ &= m.\end{aligned}$$

The slope using any two points is equal to the same constant, m. According to Definition 2.1, this means the graph of $y = mx + b$ is a straight line.

Part 2. Suppose ℓ is a straight line. Let $(x_1, y_1), (x_2, y_2)$, $x_1 \neq x_2$, be any two fixed points on ℓ and let (x, y) be any other (variable) point on ℓ. See (b) Figure 2.5 on the following page. Since ℓ is straight, the slope, m, computed using any pair of points on ℓ must equal the slope computed using any other pair of points on ℓ. This implies



(a) $y = mx + b$, m, b constant (b) ℓ is a straight line

FIGURE 2.5. Both straight lines

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{y - y_1}{x - x_1}$$

so that

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \tag{2.1}$$

but $\dfrac{y_2 - y_1}{x_2 - x_1}$ is the slope, m, so

$$y - y_1 = m(x - x_1) \tag{2.2}$$

rearranging

$$\begin{aligned} y &= mx - mx_1 + y_1, \\ &= mx + (y_1 - mx_1), \end{aligned}$$

since m, x_1 and y_1 are constants, $y_1 - mx_1$ is a constant, call it b. Thus,

$$y = mx + b. \tag{2.3}$$

Together Parts 1 and 2 prove the theorem. ■

We left hanging the discussion of the lines shown in Figure 2.3 on page 22. According Part 1 of Theorem 2.1 on the preceding page, the line $y = x$ is straight because its equation is $y = mx + b$; $m = 1, b = 0$ and the exponent on x is 1.

But what about the line $y = x^{1.01}$? According to Part 2 of Theorem 2.1, the graph of $y = x^{1.01}$ is not a straight line, because it is not the form $y = mx + b$.

The exponent on x is 1.01 not 1. We will return to this question at the end of Chapter 4 on page 95.

2.3. Various forms of the equation of a straight line.

Equation 2.1 on the facing page is called the **two point form** of the equation of a line. Although this equation looks useful for finding the equation of the line through two points, most people use Equation 2.2 instead.

Equation 2.2 is called the **point-slope form** of the equation of a line. It is handy for finding the equation of a line when the line's slope and a point on the line are known or when two points on the line are known.

Equation 2.3 is called the **slope-intercept form** of the equation of a line. It provides the slope and the y-intercept of the line by inspection.

Another form of the equation of a line that comes up often is

$$ax + by = c, \tag{2.4}$$

where neither a nor b are 0. Equation 2.4 is called the **standard form** equation of the line. When possible, we write Equation 2.4 with integer coefficients.

Yet another form of the equation for a line is Equation 2.5

$$\frac{x}{a} + \frac{y}{b} = 1, \tag{2.5}$$

where neither a nor b are 0. This is called the **intercept form** of the equation of a line. The x-intercept is a and the y-intercept is b, each obtained by inspection.

The phrase "of the equation of a line" is awkward, so instead we will usually say "of a line". For example, instead of saying "Equation 2.2 is the point-slope form of the equation of a line," we will say "Equation 2.2 is the point-slope form of a line."

The three equations following are used frequently. You should know each equation by name.

Point-slope form: $\mathbf{y - y_1 = m(x - x_1)}$.
Slope-intercept form: $\mathbf{y = mx + b}$.
Standard form: $\mathbf{ax + by = c}$ where a and b not both 0..

Where (x_1, y_1) and (x, y) are points on the line, and m, a, b and c are constants.

2.4. Applications

2.4.1. Equation from points

When you wish to know the equation of a line that passes through two particular points, the point-slope form of the line is your ally. Begin by writing the general equation $y - y_1 = m(x - x_1)$. Then your goal is to substitute the specific values of m, x_1, and y_1.

Example 2.1

Find the equation of the line through the points $(1,5), (4,9)$.

Solution

Use point-slope form of the line: $y - y_1 = m(x - x_1)$.

$$m = \frac{9-5}{4-1} = \frac{4}{3}.$$

The coordinates of either given point will do for x_1 and y_1.

$$y - 5 = \frac{4}{3}(x - 1).$$

Example 2.2

Find the equation of the line through the points $(-2,7), (3,5)$. Answer in standard form.

Solution

Use the point-slope form of the line: $y - y_1 = m(x - x_1)$.

$$m = \frac{5-7}{3+2} = \frac{-2}{5}.$$

Then

$$y - 5 = \frac{-2}{5}(x - 3) \quad \text{(point-slope form)}.$$

Rearranging,

$$\begin{aligned} 5y - 25 &= -2(x - 3) \\ 2x + 5y &= 31 \quad \text{(standard form)}. \end{aligned}$$

Example 2.3

Find the equation of the line through the points $(-2,-6), (-5,7)$. Answer in slope-intercept form.

Solution

Use point-slope form of the line: $y - y_1 = m(x - x_1)$.

$$m = \frac{7+6}{-5+2} = \frac{13}{-3} = \frac{-13}{3}.$$

Then

$$y + 6 = \frac{-13}{3}(x+2) \quad \text{(point-slope form)}.$$

Rearranging,

$$\begin{aligned} y + 6 &= \frac{-13}{3}x - \frac{26}{3} \\ y &= \frac{-13}{3}x - \frac{44}{3} \quad \text{(slope-intercept form)}. \end{aligned}$$

Example 2.4

Find the equation of the line through the points $(8,-3), (-1,9)$. Answer in standard form.

Solution

Use point-slope form of the line: $y - y_1 = m(x - x_1)$.

$$m = \frac{9+3}{-1-8} = \frac{12}{-9} = \frac{-4}{3}.$$

Then

$$y - 9 = \frac{-4}{3}(x+1) \quad \text{(point-slope form)}.$$

Rearranging,

$$\begin{aligned} 3y - 27 &= -4x - 4 \\ 4x + 3y &= 23 \quad \text{(standard form)}. \end{aligned}$$

Example 2.5

Find the equation of the line through the points $\left(\frac{1}{2}, 3\right)\left(\frac{1}{3}, \frac{3}{4}\right)$. Answer in standard form.

Solution

Use point-slope form of the line: $y - y_1 = m(x - x_1)$.

$$m = \frac{\frac{3}{4} - 3}{\frac{1}{3} - \frac{1}{2}} = \frac{\frac{-9}{4}}{\frac{-1}{6}} = \frac{27}{2}.$$

Then

$$y-3=\frac{27}{2}\left(x-\frac{1}{2}\right) \quad \text{(point-slope form).}$$

Rearranging,

$$\begin{aligned}
2y-6&=27\left(x-\frac{1}{2}\right)\\
2y-6&=27x-\frac{27}{2}\\
4y-12&=54x-27\\
54x-4y&=15 \quad \text{(standard form).}
\end{aligned}$$

Example 2.6

Find the equation of the line through the points $(0,0),(3,3)$. Answer in slope-intercept form.

Solution

Use point-slope form of the line: $y-y_1=m(x-x_1)$.

$$m=1.$$

Then

$$\begin{aligned}
y-0&=1(x-0) \quad \text{(point-slope form)}\\
y&=x.
\end{aligned}$$

Example 2.7

Write the equation in standard from of the line shown in Figure 2.6 on the next page.

Solution

Use point-slope form of the line: $y-y_1=m(x-x_1)$.

$$m=\frac{6+3}{-2-4}=-\frac{9}{6}=-\frac{3}{2}.$$

Then,

$$y-6=-\frac{3}{2}(x+2).$$

Rearranging,

$$\begin{aligned}
2y-12&=-3x-6\\
3x+2y&=6.
\end{aligned}$$

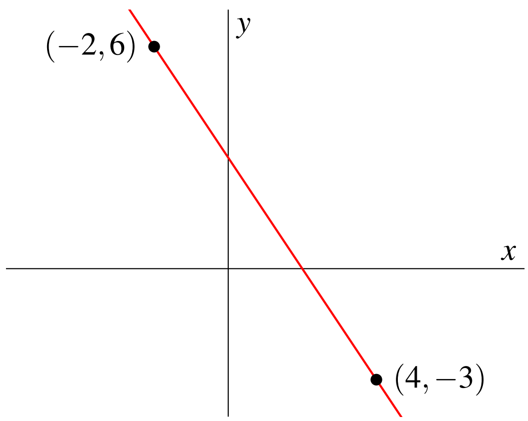


FIGURE 2.6. Line through $(-2,6)$ and $(4,-3)$.

2.4.2. Slope and intercepts

An equation in slope-intercept form needs only to be eyeballed to discover the slope and y-intercept.

If only the intercepts are needed, it hardly matters what form the equation is in. Remember that the first coordinate of the y-intercept must be 0 and the second coordinate of the x-intercept must be 0. Substituting 0 for x, then solving for y produces the y-intercept. The x-intercept is found by substituting 0 for y.

Example 2.8

Find the slope and y-intercept of $y = \frac{2}{5}x + 7$.

Solution

The equation is in slope-intercept form. By inspection the slope is $\frac{2}{5}$ and the y-intercept is 7.

Example 2.9

Find the slope and y-intercept of $2x + 3y = 6$.

Solution

Rewrite the equation in slope-intercept form.

$$\begin{aligned} 2x + 3y &= 6 \\ 3y &= -2x + 6 \\ y &= \frac{-2}{3}x + 2. \end{aligned}$$

So the slope is $\frac{-2}{3}$ and the y-intercept is 2.

Example 2.10

Find the x-intercept and the y-intercept of the line $4x+3y=7$.

Solution

The symbol "$\implies$" means "implies". "$a \implies b$" means "a implies b" or "if a then b".

$$\text{When } y=0, \quad 4x+0=7 \implies x=\frac{7}{4}.$$
$$\text{When } x=0, \quad 0+3y=7 \implies y=\frac{7}{3}.$$

So, the x-intercept is $\frac{7}{4}$ and the y-intercept is $\frac{7}{3}$.

Example 2.11

Find the slope, the y-intercept, and the x-intercept of the line $ax+by=c$.

Solution

Rewrite the equation in slope-intercept form $y=mx+b$.

$$ax+by=c \tag{2.6}$$
$$by=-ax+c$$
$$y=\frac{-a}{b}x+\frac{c}{b}.$$

By inspection, the slope is $\frac{-a}{b}$ and the y-intercept is $\frac{c}{b}$.
To get the x-intercept, let $y=0$ in Equation 2.6. Then

$$ax+0=c$$
$$x=\frac{c}{a}.$$

So, the x-intercept is $\frac{c}{a}$. ■

We can work Example 2.11 writing little or nothing other than the answer.

Example 2.12 (Example 2.11 revisited)

Find the slope, the y-intercept, and the x-intercept of the line $ax+by=c$.

Solution

For x-intercept, imagine $y=0$, solve $ax=c$ mentally to get $x=\frac{c}{a}$.

For y-intercept, imagine $x=0$, solve $by=c$ mentally to get $y=\frac{c}{b}$.

For the slope, mentally solve $ax+by=c$ for y ignoring all terms on the right hand side except the term with x. $by=-ax+\cdots$, then $y=\frac{-a}{b}x+\cdots$. So, slope is $\frac{-a}{b}$.

Exercise 2.1

Find the equation of the line through the points. See Examples 2.1-2.7.

Answer in point-slope form.

1. $(3,8),(12,14)$.
2. $(4,-1),(12,-7)$.
3. $(-7,-2),(14,-5)$.
4. $(-3,15),(4,-6x)$.
5. $(-2,-13),(4,11)$.
6. $(0,1),(12,-27)$.

Answer in slope-intercept form.

7. $(-20,2),(-5,4)$.
8. $(-4,-4),(8,-1)$.
9. $(-3,25),(4,17)$.
10. $(-4,9),(6,4)$.
11. $(0,2),(12,11)$.
12. $(-9,-2),(-3,-6)$.

Answer in standard form.

13. $(6,-29),(4,-17)$.
14. $(5,-9),(7,-13)$.
15. $(0,-4),(2,3)$.
16. $(2,-3),(-3,-2)$.
17. $(-5,-5),(3,2)$.
18. $(-4,5),(0,-2)$.
19. $(-5,0),(-4,4)$.
20. $(1,-3),(0,4)$.
21. $(-5,2),(3,-5)$.
22. $(-1,-3),(-5,2)$.
23. $(0,4),(-4,1)$.
24. $(5,0),(2,2)$.

Find the slope and both intercepts. See Examples 2.8-2.12.

25. $2x+5y=10$.
26. $10x+6y=-30$.
27. $2x+5y=-10$.
28. $x+7y=7$.
29. $4x+5y=20$.
30. $11x-11y=11$.
31. $3x-2y=6$.
32. $2x-3y=6$.
33. $3x+2y=2$.
34. $5x-2y=2$.
35. $7x+21y=-7$.
36. $3x+10y=-2$.
37. $7x+2y=-14$.
38. $5x-y=-5$.
39. $12x+13y=156$.
40. $2x+y=-4$.

2.4.3. Parallel lines

Example 2.13

Find the equation of the line ℓ that is parallel to the line $y = \frac{3}{5}x - 4$ and passes through the point $(-5, 11)$. Answer in standard form.

Solution

Use point-slope form of the line: $y - y_1 = m(x - x_1)$. Since ℓ is parallel to $y = \frac{3}{5}x - 4$ it must have the same slope. That slope is, by inspection, $\frac{3}{5}$. Then,

$$\begin{aligned} y - 11 &= \frac{3}{5}(x+5) \\ 5y - 55 &= 3(x+5) \\ 5y - 55 &= 3x + 15 \\ 3x - 5y &= -70. \end{aligned}$$

2.4.4. Perpendicular lines

The slopes of perpendicular lines are related according to Theorem 2.2, but the proof must wait until the student has learned school geometry.

Theorem 2.2
Let ℓ_1 have slope m_1 and ℓ_2 have slope m_2. Lines ℓ_1 and ℓ_2 are perpendicular if and only if $m_1 m_2 = -1$.

Example 2.14

Show that the lines $\ell_1 : 2x + 3y = 1$ and $\ell_2 : 3x - 2y = 1$ are perpendicular.

Solution

The slope of ℓ_1 is $-\frac{2}{3}$. The slope of ℓ_2 is $\frac{3}{2}$. The product of the slopes, $m_1 m_2$, is $-\frac{2}{3} \cdot \frac{3}{2} = -1$. By Theorem 2.2, the lines are perpendicular.

Example 2.15

Find the line ℓ through point $P(0, -4)$ that is perpendicular to line $\ell_1 : 3x + 2y = 6$. See Figure 2.7 on the facing page.

Solution

Use the point-slope form, $y - y_1 = m(x - x_1)$. Since $P(0, -4)$ is on ℓ, we immediately have $y + 4 = mx$. All that remains it to know m. Since the lines are perpendicular, Theorem 2.2 says that the product of the slopes is -1. The

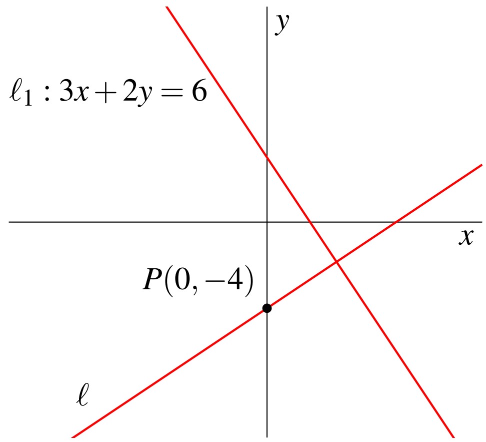


FIGURE 2.7. Lines ℓ_1 and ℓ are perpendicular.

slope of ℓ_1 is $-\frac{3}{2}$, so

$$-\frac{3}{2}m = -1$$

which means

$$m = \frac{2}{3}.$$

Therefore, the equation of the line through $P(0,-4)$ that is perpendicular to line $\ell_1 : 3x+2y=6$ is

$$y+4 = \frac{2}{3}x \iff 2x-3y=12.$$

2.4.5. Intersecting lines

Example 2.16

Find the point at which lines ℓ_1 and ℓ_2 intersect if the equation for line ℓ_1 is $2x+3y=-6$ and the equation of line ℓ_2 is $x-3y=6$.

Solution

The point of intersection must be a point on each of the two lines. So, its coordinates must make each equation true. The coordinates are easily found by solving for x and y.

$$\begin{bmatrix} 2x+3y=-6 \\ x-3y=6 \end{bmatrix}.$$

The point of intersection is found to be $(0,-2)$. Figure 2.8 on the following page.

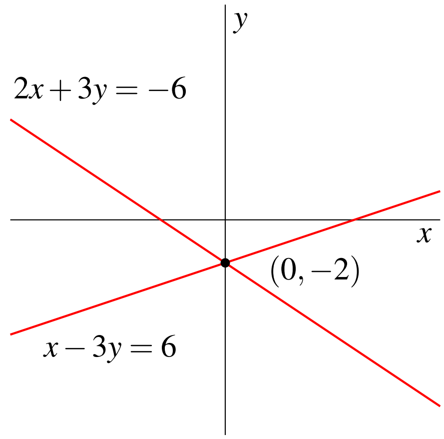


FIGURE 2.8. The point of intersection of two lines must be on each line.

Example 2.17

Find the point at which lines ℓ_1 and ℓ_2 intersect if the equation for line ℓ_1 is $y=\frac{2}{3}x+2$ and the equation of line ℓ_2 is $y=2x-5$.

Solution

The point of intersection must be a point on each of the two lines. So, its coordinates must make each equation true. Since $\frac{2}{3}x+2$ and $2x-5$ both equal y,

$$\begin{aligned}\frac{2}{3}x+2&=2x-5\\2x+6&=6x-15\\4x&=21\\x&=\frac{21}{4}.\end{aligned}$$

When $x=\frac{21}{4}$, $y=2\left(\frac{21}{4}\right)-5=\frac{11}{2}$. So the point of intersection is $\left(\frac{21}{4},\frac{11}{2}\right)$.

Example 2.18

Find the point at which lines ℓ_1 and ℓ_2 intersect if the equation for line ℓ_1 is $y=\frac{11}{19}x+2$ and the equation of line ℓ_2 is $\frac{11}{19}x+1$.

Solution

Look and think before you leap. The lines have the same slope. They are two different lines, because ℓ_1 goes through $(0,2)$ but ℓ_2 goes through $(0,1)$.

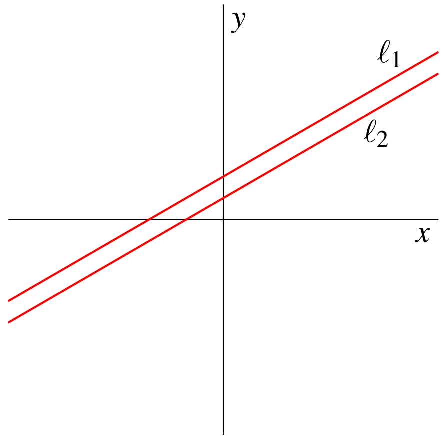


FIGURE 2.9. Parallel lines have no point in common.

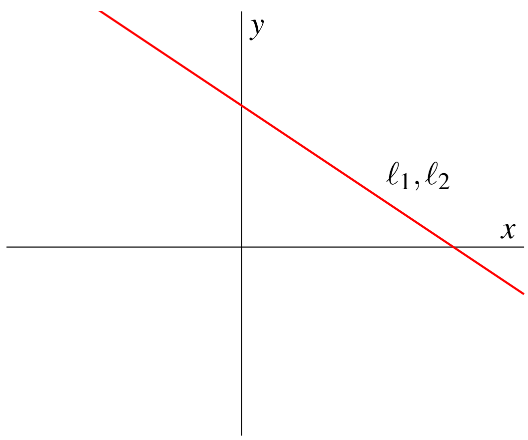


FIGURE 2.10. "ℓ_1", and "ℓ_2" name the same line.

The lines are parallel, so cannot intersect. Answer: The point of intersection does not exist. Figure 2.9.

Example 2.19

Find the point at which lines ℓ_1 and ℓ_2 intersect if the equation for line ℓ_1 is $2x+3y=18$ and the equation of line ℓ_2 is $4x+6y=36$.

Solution

Notice that there is only one equation, because $4x+6y=36 \iff 2x+3y=18$. Answer: Every point on line $2x+3y=18$ is a point of intersection. Figure 2.10.

Example 2.20

Let ℓ_1 be the line $7x+2y=13$ and let ℓ_2 be the line $2x-y=-1$. (a) Find the

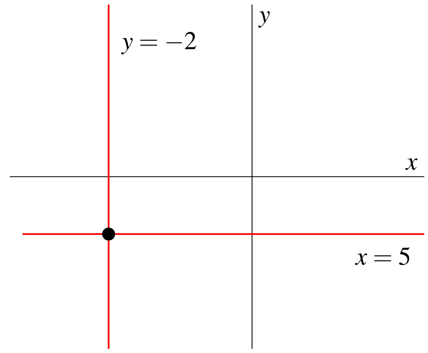


FIGURE 2.11. The lines $x = 5$ and $y = -2$.

point of intersection of ℓ_1 and ℓ_2. (b) Determine if ℓ_1 and ℓ_2 are perpendicular at their point of intersection.

Solution

(a) To find the point of intersection, solve

$$\begin{bmatrix} 7x + 2y = 13 \\ 2x - y = -1 \end{bmatrix}.$$

The point of intersection is $(1,3)$.

(b) The slope of ℓ_1 is $-\frac{7}{2}$. The slope of ℓ_2 is 2. Since $-\frac{7}{2} \cdot 2 = -7 \neq -1$, The lines are not perpendicular.

Example 2.21

Find the point of intersection of the lines $x = 5$ and $y = -2$.

Solution

The line $x = 5$ is parallel to the y-axis and the line $y = -2$ is parallel to the x-axis. The lines intersect at the point $(5, -2)$. Imagine the graph in Figure 2.11.

Exercise 2.2

Find the line through point P and parallel to line ℓ. Answer in standard form.

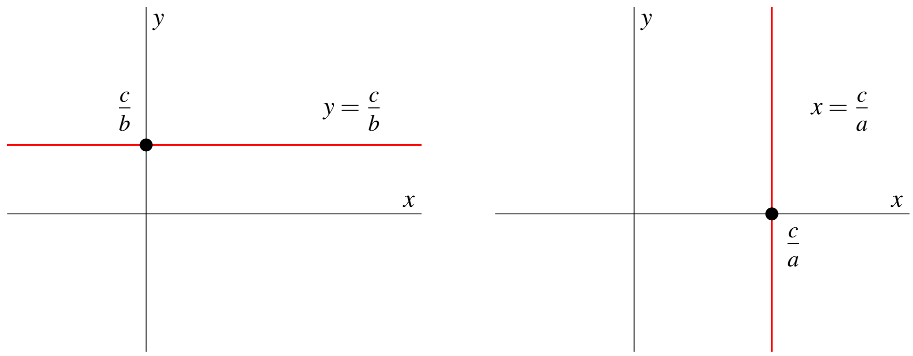
1. $P(4,-2)$, $\ell: y=-\frac{1}{4}x+4$.
2. $P(0,-5)$, $\ell: y=6x-4$.
3. $P(4,2)$, $\ell: y=x-1$.
4. $P(-1,0)$, $\ell: y=-3x-2$.
5. $P(4,4)$, $\ell: y=x+2$.
6. $P(5,2)$, $\ell: y=\frac{7}{5}x+2$.
7. $P(-1,3)$, $\ell: y=-8x-2$.
8. $P(-4,-3)$, $\ell: y=x-4$.
9. $P(2,3)$, $\ell: y=3x$.
10. $P(-5,3)$, $\ell: x=0$.

Find the line through point P and perpendicular to line ℓ. Answer in standard form.

11. $P(-3,5)$, $\ell: y=\frac{3}{8}x-2$.
12. $P(5,1)$, $\ell: y=\frac{5}{3}x-2$.
13. $P(2,-3)$, $\ell: y=2x-3$.
14. $P(-2,2)$, $\ell: y=-\frac{5}{3}x-5$.
15. $P(3,-3)$, $\ell: y=\frac{5}{8}x-5$.
16. $P(-2,5)$, $\ell: y=\frac{1}{3}x-3$.
17. $P(4,5)$, $\ell: y=-\frac{9}{10}x-2$.
18. $P(1,3)$, $\ell: y=-\frac{3}{7}x+1$.
19. $P(-2,3)$, $\ell: y=\frac{1}{2}x+4$.
20. $P(-2,5)$, $\ell: y=-2x-5$.

Find the line perpendicular to line ℓ at point P. Answer in standard form.

21. $\ell: x-2y=-2$, $P(-4,-1)$
22. $\ell: 2x-5y=-10$, $P(5,4)$



(a) $ax + by = c$ when $a = 0$. (b) $ax + by = c$ when $b = 0$.

FIGURE 2.12. Degenerate cases of $ax + by = c$.

2.4.6. Degenerate cases

On page 39, we said the standard form of a line is $ax + by = c$ where a and b not both 0. What happens if either of a or b is 0?

If $a = 0$ and $b \neq 0$, the equation of the line is $by = c$ or, what is equivalent, $y = \frac{c}{b}$. This is a line parallel to the x-axis through the point $(0, \frac{c}{b})$ on the y-axis. Figure 2.12.

If $b = 0$ and $a \neq 0$, the equation of the line is $ax = c$, or, what is equivalent, $x = \frac{c}{a}$. This is a line parallel to the y-axis through the point $(\frac{c}{a}, 0)$ on the x-axis. Figure 2.13 on the next page.

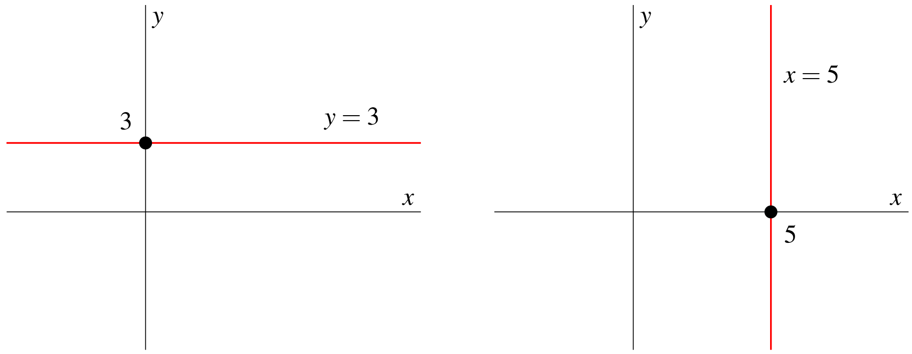
You might be wondering "When I see '$y = 5$', how am I supposed to know whether this is assigning the value 5 to the letter y *or* indicating the line through 5 on the y-axis?" The context will make the meaning clear. If you write "the line $y = 5$", you will certainly remove any doubt.

Example 2.22

(a) Graph the line $y = 3$, and (b) graph the line $x = 5$.

Solution

See Figure 2.13 on the facing page.



(a) $y = 3$. (b) $x = 5$.

FIGURE 2.13. Lines parallel to the axes.

2.5. Summary

Of the several forms of the equation for the straight line, the three listed below are the ones you must know and understand. You should know the following equations by their names.

Point-slope form: $\mathbf{y - y_1 = m(x - x_1)}$.

Slope-intercept form: $\mathbf{y = mx + b}$.

Standard form: $\mathbf{ax + by = c}$ where a and b not both 0..

Ideally, you should be able to derive the point-slope equation of a line, given that the line is a straight line.

Some of the questions in Exercise 2.3 on the next page are hard. Complete solutions, not just answers, to Exercise 2.3 are in the back of the book. Enjoy this exercise as an exploration of some of the ideas you met in this chapter.

Exercise 2.3

The letter m represents a constant whenever it appears in the questions below.

1. Without using Theorem 2.1 on page 23, show that the graph of $y = mx^2 + b, m \neq 0$, is not a straight line. [Hint: find somewhere on the graph that is not straight.]

2. Show that nowhere is the graph of $y = mx^2 + b, m \neq 0$, straight. This question fits too well here not to mention it. But we postpone it. It will turn up as Problem 5 on page 220.

3. The proof of Theorem 2.1 on page 23 showed the graph of $y = mx + b$ is a straight line. Try to use a similar argument to show that graph of the function $y = mx^2 + b, m \neq 0$, is a straight line and note where the argument fails.

4. In question #1, you probably showed that the slope of $y = mx^2 + b$ is not constant, then concluded that the graph of $y = mx^2 + b$ cannot be a straight line. But, you would have begun by assuming that m is constant. How can it be that m is constant, but the slope of $y = mx^2 + b$ is *not* constant?

5. On on page 24, we replaced, $y_1 - mx_1$ with a single constant b. What justifies our doing that?

6. Is the sum of two linear functions a linear function? Give a reason for your answer.

7. Theorem 2.1 on page 23 limits itself to non-vertical lines. Why?

8. Use Theorem 2.1 to prove
   a) that $y = x$ is a straight line.
   b) that $y = x^{1.01}$ is not a straight line.

Chapter 3

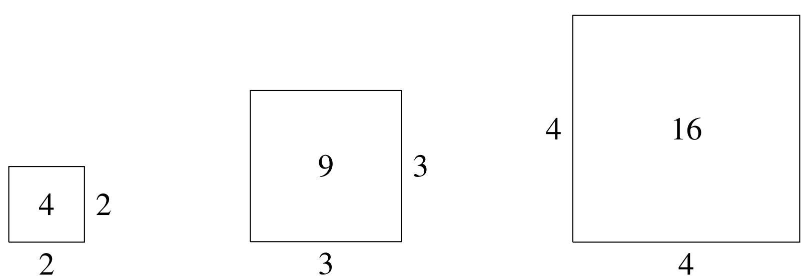
Radicals

3.1. Square numbers

Whoever first noticed that the numbers $4, 9, 16, 25, 36, 49, 64, 81, 100$ all had a quality in common must have been delighted by the discovery. The common quality is that each of these numbers is the product a factor used exactly twice. For example,

$$\begin{aligned} 4 &= 2 \cdot 2, \\ 9 &= 3 \cdot 3, \\ 16 &= 4 \cdot 4, \\ &\vdots \\ 100 &= 10 \cdot 10. \end{aligned}$$

The numbers $4, 9, 16, 25, 36, 49, 64, 81, 100, \cdots$ are called "square numbers". The name is descriptive. The area of a square of side length 2 is 4, of side length 3, 9, and so on.



(a) 2 by 2 (b) 3 by 3 (c) 4 by 4

FIGURE 3.1. Squares of side lengths $2, 3$ and 4.

Square numbers up to 144 live on the diagonal $1 \cdots 12$ of the 12×12 multiplication table. This diagonal is also a line of symmetry for Table 3.1.

1	2	3	4	5	6	7	8	9	10	11	12
2	**4**	6	8	10	12	14	16	18	20	22	24
3	6	**9**	12	15	18	21	24	27	30	33	36
4	8	12	**16**	20	24	28	32	36	40	44	48
5	10	15	20	**25**	30	35	40	45	50	55	60
6	12	18	24	30	**36**	42	48	54	60	66	72
7	14	21	28	35	42	**49**	56	63	70	77	84
8	16	24	32	40	48	56	**64**	72	80	88	96
9	18	27	36	45	54	63	72	**81**	90	99	108
10	20	30	40	50	60	70	80	90	**100**	110	120
11	22	33	44	55	66	77	88	99	110	**121**	132
12	24	36	48	60	72	84	96	108	120	132	**144**

TABLE 3.1. 12×12 multiplication table

3.2. Square roots

When a number is the product of exactly one positive factor used twice, the factor is called the square root of that number. For example,

$$2 \cdot 2 = 4 \text{ so 2 is the square root of 4,}$$
$$3 \cdot 3 = 9 \text{ so 3 is the square root of 9,}$$
$$4 \cdot 4 = 16 \text{ so 4 is the square root of 16.}$$

We may as well have a definition.

Definition 3.1 (Square root)
Let b be any positive number. The **square root** of b is the positive number $\sqrt{b}$, if $\sqrt{b} \cdot \sqrt{b} = b$. ■

For example,

$$2 = \sqrt{4}, \text{ because 2 is positive and } 2 \cdot 2 = 4.$$
$$3 = \sqrt{9}, \text{ because 3 is positive and } 3 \cdot 3 = 9.$$
$$4 = \sqrt{16}, \text{ because 4 is positive and } 4 \cdot 4 = 16.$$

Even though $(-5)(-5) = 25$, we do not write "$-5 = \sqrt{25}$", because -5 is not a positive number.

The additive inverse of the number $\sqrt{b}$, is written $-\sqrt{b}$. For example, the additive inverse of the square root of 49 is $-\sqrt{49}$ which equals -7.

Example 3.1

For $n = 16, 25, 36$, (a) find $\sqrt{n}$ and (b) find $-\sqrt{n}$.

Solution.

$$\begin{array}{llll} \text{For } n = 16, & \text{(a)} & \sqrt{16} = 4, & (b) \quad -\sqrt{16} = -4. \\ \text{For } n = 25, & \text{(a)} & \sqrt{25} = 5, & (b) \quad -\sqrt{25} = -5. \\ \text{For } n = 36, & \text{(a)} & \sqrt{36} = 6, & (b) \quad -\sqrt{36} = -6. \end{array}$$

Example 3.2

For $n = 16, 25, 36$, find all numbers that used as a factor twice produce n.

Solution.

$$\begin{array}{ll} \text{For } n = 16, & \{4, -4\}. \\ \text{For } n = 25, & \{5, -5\}. \\ \text{For } n = 36, & \{6, -6\}. \end{array}$$

■

The question of Example 3.2 could have been asked this way: "Find all numbers whose square is *n*."

3.2.1. No square roots of negative numbers

It is easy to see that a negative number does not have a square root in the real numbers. After all, what could the square root be? Not 0, because $0 \cdot 0$ is not negative. Not a positive number, since positive · positive is positive. But not negative, because negative · negative is positive. There are no other possibilities. This is why in Definition 3.1 on the facing page we specify that b is a positive number.

3.2.2. Square roots of rational numbers

A few examples will give you the idea.

Example 3.3

$$\sqrt{\frac{1}{16}} = \frac{1}{4}, \text{ because } \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}.$$

$$\sqrt{\frac{1}{25}} = \frac{1}{5}, \text{ because } \frac{1}{5} \cdot \frac{1}{5} = \frac{1}{25}.$$

$$\sqrt{\frac{4}{25}} = \frac{2}{5}, \text{ because } \frac{2}{5} \cdot \frac{2}{5} = \frac{4}{25}.$$

$$\sqrt{\frac{9}{49}} = \frac{3}{7}, \text{ because } \frac{3}{7} \cdot \frac{3}{7} = \frac{9}{49}.$$

Example 3.4

$$\sqrt{\frac{1}{100}} = \frac{1}{10}$$

$$\sqrt{\frac{1}{10000}} = \frac{1}{100}$$

$$\sqrt{\frac{1}{1,000,000}} = \frac{1}{1000}$$

■

When the rational number is written as a decimal, it is often helpful to view the number as a fraction. Several examples follow.

Example 3.5

$$\sqrt{0.09} = \sqrt{\frac{9}{100}} = \frac{3}{10} = 0.3.$$

$$\sqrt{0.0004} = \sqrt{\frac{4}{10000}} = \frac{2}{100} = 0.02.$$

$$\sqrt{1.44} = \sqrt{\frac{144}{100}} = \frac{12}{10} = 1.2.$$

Usually we answer with a decimal if the question is posed as a decimal.

3.2.3. Non-obvious square roots

Simplify $\sqrt{625}$. You might happen to know that $625 = 25 \cdot 25$. If so, you will immediately answer $\sqrt{625} = 25$. But, what if you do not just happen to know $625 = 25 \cdot 25$?

In a moment we will see that prime factorization comes to the rescue. But first, we must note that the square root of a pair of factors is one of the pair. For example,

$$\sqrt{2\cdot 2} = 2.$$
$$\sqrt{8\cdot 8} = 8.$$
$$\sqrt{2\cdot 2\cdot 8\cdot 8} = 2\cdot 8 = 16.$$
$$\sqrt{5\cdot 5\cdot 11\cdot 11} = 5\cdot 11 = 55.$$
$$\sqrt{2\cdot 2\cdot 2\cdot 2} = 2\cdot 2 = 4.$$
$$\sqrt{7\cdot 7\cdot 7\cdot 7} = 7\cdot 7 = 49.$$
$$\sqrt{7\cdot 7\cdot 7\cdot 7\cdot 2\cdot 2} = 7\cdot 7\cdot 2 = 98.$$
$$\sqrt{7\cdot 7\cdot 3\cdot 3\cdot 5\cdot 5} = 7\cdot 3\cdot 5 = 105.$$

Once the prime factorization of a number is known, finding the square root of the number is no harder than spotting pairs of factors.

Example 3.6

Simplify $\sqrt{213444}$.

Solution

We rewrite 213444 as a product of prime factors:

$$213444 = 2\cdot 2\cdot 3\cdot 3\cdot 7\cdot 7\cdot 11\cdot 11.$$

So,

$$\begin{aligned}\sqrt{213444} &= \sqrt{2\cdot 2\cdot 3\cdot 3\cdot 7\cdot 7\cdot 11\cdot 11}\\ &= 2\cdot 3\cdot 7\cdot 11\\ &= 462.\end{aligned}$$

Example 3.7

Simplify $\sqrt{30}$.

Solution

We rewrite 30 as a product of prime factors:

$$30 = 2\cdot 3\cdot 5.$$

The number 30 has no repeated prime factors. It cannot be simplified further.

Exercise 3.1

Simplify each square root.

1. $\sqrt{25}$
2. $\sqrt{121}$
3. $\sqrt{64}$
4. $\sqrt{36}$
5. $\sqrt{100}$
6. $\sqrt{10000}$
7. $\sqrt{169}$
8. $\sqrt{-4}$
9. $\sqrt{\frac{4}{25}}$
10. $\sqrt{\frac{64}{121}}$
11. $\sqrt{\frac{25}{81}}$
12. $\sqrt{\frac{144}{9}}$
13. $\sqrt{0.36}$
14. $\sqrt{0.49}$
15. $\sqrt{0.04}$
16. $\sqrt{0.09}$
17. $\sqrt{0.01}$
18. $\sqrt{0.0001}$
19. $\sqrt{0.0036}$

Write the numbers whose square is the number given

20. 4
21. 25
22. 81
23. 144
24. 49
25. $\frac{9}{49}$
26. $\frac{64}{25}$

Simplify

27. $\sqrt{1764}$
28. $\sqrt{1225}$
29. $\sqrt{1936}$
30. $\sqrt{8281}$
31. $\sqrt{2025}$
32. $\sqrt{15625}$
33. $\sqrt{256}$
34. $\sqrt{729}$

3.2.4. Numbers that are not square numbers

Suppose we wish to simplify $\sqrt{50}$. It seems our luck has run out, because 50 is between square numbers $49 = 7^2$ and $64 = 8^2$. So, out of luck? Almost, but not quite. We can simplify $\sqrt{50}$ to a certain extent. Example 3.8 shows how.

Example 3.8

Simplify $\sqrt{50}$.

Solution.

$$\begin{aligned}\sqrt{50} &= \sqrt{2 \cdot 5 \cdot 5} \\ &= 5\sqrt{2}.\end{aligned}$$

Note that $2 \cdot 5 \cdot 5$ is a pair of 5's with a 2 left over.

Example 3.9

Simplify $\sqrt{27}$.

Solution.

$$\begin{aligned}\sqrt{27} &= \sqrt{3 \cdot 3 \cdot 3} \\ &= 3\sqrt{3}.\end{aligned}$$

Note that $3 \cdot 3 \cdot 3$ is a pair of 3's with a 3 left over.

Example 3.10

Simplify $\sqrt{8400}$.

Solution.

$$\begin{aligned}\sqrt{8400} &= \sqrt{2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 \cdot 5 \cdot 7} \\ &= 2 \cdot 2 \cdot 5\sqrt{3 \cdot 7} \\ &= 20\sqrt{21}. \quad \blacksquare\end{aligned}$$

Writing multiple appearances of a factor using exponents makes the working tidy.

Example 3.11

Simplify $\sqrt{98784}$.

Solution

$$\begin{aligned}\sqrt{98784} &= \sqrt{2^5 3^2 7^3} \\ &= \sqrt{2^4 \cdot 2 \cdot 3^2 \cdot 7^2 \cdot 7}.\end{aligned}$$

2^4 is 2 pairs of 2, 3^2 is one pair of 3, and 7^2 is one pair of 7.

$$\begin{aligned} &= 2^2 \cdot 3 \cdot 7\sqrt{2 \cdot 7} \\ &= 84\sqrt{14}. \quad \blacksquare\end{aligned}$$

Prime factorization is a tool. Use it when it is helpful. In the next example, you might not need that tool.

Example 3.12

Simplify $\sqrt{1600}$.

Solution

$$\begin{aligned}\sqrt{1600} &= \sqrt{16 \cdot 100} \\ &= 4 \cdot 10 \\ &= 40.\end{aligned}$$

Exercise 3.2

Simplify.

1. $\sqrt{12}$
2. $\sqrt{45}$
3. $\sqrt{175}$
4. $\sqrt{96}$
5. $\sqrt{1000}$
6. $\sqrt{1372}$
7. $\sqrt{216}$
8. $\sqrt{8}$
9. $\sqrt{1728}$
10. $\sqrt{27}$
11. $\sqrt{20}$
12. $\sqrt{250}$
13. $\sqrt{490}$
14. $\sqrt{490000}$
15. $\sqrt{160000}$
16. $\sqrt{504}$

3.2.5. The square root of 2

On page 47 we considered $\sqrt{50}$ and found that we could simplify it to a certain extent, $\sqrt{50} = 5\sqrt{2}$. We might wonder if $\sqrt{2}$ can be simplified. Perhaps there is some rational number that equals the square root of 2. Do not bother trying to find it, because you will fail. It is not that you are insufficiently clever. You will not find such a rational number because it does not exist. We can prove that. First, we prove a theorem we will need.

Theorem 3.1
Let a and b be two positive real numbers. If $a > b$ then $\sqrt{a} > \sqrt{b}$.

Proof. Let a and b be two positive real numbers. Suppose that $a > b$. Then,

$$a - b > 0$$

$$(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b}) > 0 \tag{3.1}$$

Writing $a - b$ as $(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b})$ may seem clever. After Chapter 5, you will think of that right away.

$(\sqrt{a} + \sqrt{b})$ is positive, because it is the sum of positive numbers. So,

$$\sqrt{a} - \sqrt{b} > 0. \tag{3.2}$$

This means

$$\sqrt{a} > \sqrt{b}.$$

■

Inequality 3.1 is essential. But does $a - b$ equal $(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b})$? The answer is "yes". Here is why.

$$\begin{aligned}(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b}) &= (\sqrt{a} - \sqrt{b})(\sqrt{a}) + (\sqrt{a} - \sqrt{b})(\sqrt{b}) \\ &= \sqrt{a}\sqrt{a} - \sqrt{b}\sqrt{a} + \sqrt{a}\sqrt{b} - \sqrt{b}\sqrt{b} \\ &= \sqrt{a}\sqrt{a} - \sqrt{b}\sqrt{b} \\ &= a - b.\end{aligned}$$

Distribution was used to good advantage.

Theorem 3.2
$\sqrt{2}$ is not a rational number.

Proof. Since $1 < 2 < 4$, Theorem 3.1 says that $\sqrt{1} < \sqrt{2} < \sqrt{4}$. This means $1 < \sqrt{2} < 2$. Suppose there is a rational number between 1 and 2 whose square is 2. Call that number r. Now, the denominator of r cannot be 1, because r is not an integer. Furthermore, r, like all rational numbers, can be fully simplified. This means the numerator and denominator of r share no common factor. According to the Fundamental Theorem of Arithmetic, each of the numerator and denominator of r can be written as a unique product of prime numbers. That is,

$$r = \frac{p_1 p_2 p_3 \cdot \cdots \cdot p_m}{q_1 q_2 q_3 \cdot \cdots \cdot q_n}$$

where the p's and q's are prime numbers and no one of the p's is equal to any of the q's. Squaring r produces

$$r^2 = r \cdot r = \frac{(p_1 p_2 p_3 \cdot \cdots \cdot p_m)(p_1 p_2 p_3 \cdot \cdots \cdot p_m)}{(q_1 q_2 q_3 \cdot \cdots \cdot q_n)(q_1 q_2 q_3 \cdot \cdots \cdot q_n)}.$$

Since no cancellations are possible, the denominator of $r^2 \neq 1$. Thus, r^2 is not an integer so it cannot equal 2. Therefore, there is no rational number equal to $\sqrt{2}$. ■

3.2.6. Irrational numbers

The number $\sqrt{2}$ is called an "irrational" number. Definition 3.2 does the honors.

Definition 3.2 (Irrational number)
A number that is not a rational number is called an **irrational** number. ■

Pretty great definition, huh? The author admits that Definition 3.2 is too contrived to be intellectually satisfying. We can do better a few years from now.

Since Definition 3.2 may have put the reader in a critical mood, the author is somewhat worried the reader will think "Great, now I know everything about the square root of 2 except why such a thing should exist. Maybe the square root of two lives with the unicorns."

Your heart is in the right place when you demand a proof that $\sqrt{2}$ exists. Just because we can utter a sound that appears to be a name for an object, does not mean that the utterance is in fact a name. There is a proof that $\sqrt{2}$ exists. But, for the next few years, you will have to take it on faith that $\sqrt{2}$ exists.

We have discussed only one irrational number. But it is easy to imagine similar proofs for $\sqrt{3}, \sqrt{5}, \sqrt{10}$ and others. In Section 3.4 on page 67 you will discover more irrational numbers called "nth" roots. For example, the third root of 8 written $\sqrt[3]{8}$.

3.2.7. The real numbers

With our discovery of the irrational numbers, we can finally be a little clearer about the real numbers. The set of real numbers is the union of the set of rational numbers and the set of irrational numbers. Figure 2.1 on page 21.

3.2.8. More irrational numbers

In order not to mislead you, we mention that the square roots you now know about and the nth roots you will learn about in Section 3.4 on page 67 are called "algebraic" numbers. The algebraic numbers are not all of the irrational numbers. There is another kind of irrational number, called "non-algebraic" or "transcendental". The number π is one example of a transcendental number. The 1-1 correspondence between the real numbers and the points on a continuous line requires the transcendental numbers. You will meet transcendental numbers in a future course.

3.3. Arithmetic with square roots

We discovered square roots. The next questions are How to add, subtract, multiply, and divide using them?

3.3.1. Addition (subtraction)

Think of $3\sqrt{2}$ and $5\sqrt{2}$ as like terms. As such, they may be combined by addition or subtraction. For example,

$$3\sqrt{2}+5\sqrt{2}=8\sqrt{2},$$

and

$$3\sqrt{2}-5\sqrt{2}=-2\sqrt{2}.$$

The explanation we gave in *Beginning Algebra* for combining like terms, works just as well with square roots.

$$\begin{aligned}3\sqrt{2}+5\sqrt{2}&=\sqrt{2}(3+5)\\&=8\sqrt{2}.\end{aligned}$$

At first sight, it may appear that $\sqrt{20}$ and $\sqrt{45}$ are unlike terms. But, when we rewrite $\sqrt{20}$ as $2\sqrt{5}$ and $\sqrt{45}$ as $3\sqrt{5}$, we see they are like terms.

$$\begin{aligned}\sqrt{20}+\sqrt{45}&=2\sqrt{5}+3\sqrt{5}\\&=5\sqrt{5}.\end{aligned}$$

Example 3.13

Simplify if possible. If the expression is already simplified, say so.

(1) $\sqrt{7}+\sqrt{7}$

(2) $2\sqrt{10}-9\sqrt{10}$

(3) $7\sqrt{5}+3\sqrt{5}$

(4) $5\sqrt{7}+5\sqrt{3}$

(5) $\sqrt{27}+\sqrt{75}$

(6) $\sqrt{28}+\sqrt{12}$

(7) $\sqrt{8}+\sqrt{98}$

(8) $3\sqrt{5}-8\sqrt{5}+2\sqrt{5}+\sqrt{16}$

Solution.

(1) $2\sqrt{7}$

(2) $-7\sqrt{10}$

(3) $10\sqrt{5}$

(4) Already simplified

(5) $3\sqrt{3}+5\sqrt{3}=8\sqrt{3}$

(6) $2\sqrt{7}+2\sqrt{3}$

(7) $2\sqrt{2}+7\sqrt{2}=9\sqrt{2}$

(8) $-3\sqrt{5}+4$

Exercise 3.3

Simplify, if possible, each expression.

1. $\sqrt{27}+\sqrt{3}$
2. $\sqrt{24}+\sqrt{24}$
3. $\sqrt{18}+\sqrt{8}$
4. $\sqrt{12}+\sqrt{27}$
5. $\sqrt{6}+\sqrt{54}$
6. $\sqrt{20}+\sqrt{5}$
7. $3\sqrt{2}-4\sqrt{18}$
8. $3\sqrt{3}-2\sqrt{12}$
9. $-4\sqrt{24}+3\sqrt{6}$
10. $4\sqrt{10}+3\sqrt{10}$
11. $3\sqrt{24}-\sqrt{96}$
12. $-\sqrt{2}-4\sqrt{50}$
13. $3\sqrt{75}+5\sqrt{12}$
14. $5\sqrt{150}+3\sqrt{-6}$
15. $-3\sqrt{20}-5\sqrt{80}$
16. $5\sqrt{90}-2\sqrt{40}$
17. $-4\sqrt{20}+5\sqrt{45}+2\sqrt{6}$
18. $-\sqrt{20}-5\sqrt{54}+3\sqrt{6}$
19. $4\sqrt{7}-4\sqrt{7}-5\sqrt{7}$
20. $3\sqrt{7}+4\sqrt{63}-2\sqrt{18}$

3.3.2. Multiplication

The product of square roots is found using Theorem 3.3.

Theorem 3.3 (Multiplication rule)
Let a and b be positive numbers. Then $\sqrt{a}\cdot\sqrt{b}=\sqrt{a\cdot b}$.

Proof. Since multiplication in the real numbers is closed, there is a number, call it x, such that $\sqrt{a}\cdot\sqrt{b}=x$. Then,

$$\begin{aligned}\sqrt{a}\cdot\sqrt{b}&=x\\(\sqrt{a}\cdot\sqrt{b})^2&=x^2\\\sqrt{a}\cdot\sqrt{b}\cdot\sqrt{a}\cdot\sqrt{b}&=x^2\\a\cdot b&=x^2\\\sqrt{a\cdot b}&=x.\end{aligned}$$

This means $\sqrt{a}\cdot\sqrt{b}=\sqrt{a\cdot b}$. ■

Example 3.14

Multiply $\sqrt{2}\sqrt{7}$.

Solution.

$$\sqrt{2}\sqrt{7}=\sqrt{2\cdot 7}=\sqrt{14}.$$

Example 3.15

Write the product of $\sqrt{2}\sqrt{5}\sqrt{13}$.

Solution.

$$\begin{aligned}\sqrt{2}\sqrt{5}\sqrt{13}&=\sqrt{2\cdot 5\cdot 13}\\&=\sqrt{130}.\end{aligned}$$

Example 3.16

Multiply $\sqrt{3}\sqrt{6}\sqrt{10}$.

Solution.

$$\begin{aligned}\sqrt{3}\sqrt{6}\sqrt{10}&=\sqrt{3\cdot 6\cdot 10}\\&=\sqrt{3\cdot 3\cdot 2\cdot 2\cdot 5}\\&=3\cdot 2\sqrt{5}=6\sqrt{5}.\end{aligned}$$ ■

Notice in Example 3.16 computing $3 \cdot 3 \cdot 2 \cdot 2 \cdot 5 = 180$ would have been wasted effort. To simplify $\sqrt{180}$ requires we see 180 as the product of factors we would have just replaced with 180. Why do work only to undo the work?

Example 3.17

Multiply $\sqrt{27}\sqrt{144}$.

Solution.

$$\begin{aligned}\sqrt{27}\sqrt{144} &= 3\sqrt{3} \cdot 12 \\ &= 36\sqrt{3}.\end{aligned}$$

■

Example 3.17 makes an important point. Simplifying each square root first, beats working with $\sqrt{27 \cdot 144}$.

Example 3.18

Simplify $5\sqrt{18} \cdot 2\sqrt{75}$.

Solution

$$\begin{aligned}5\sqrt{18} \cdot 2\sqrt{75} &= 15\sqrt{2} \cdot 10\sqrt{3} \\ &= 150 \cdot \sqrt{2} \cdot \sqrt{3} \\ &= 150\sqrt{6}.\end{aligned}$$

Example 3.19

Simplify $-3\sqrt{20} \cdot \sqrt{27}$.

Solution

$$-3\sqrt{20} \cdot \sqrt{27} = -6\sqrt{5} \cdot 3\sqrt{3} = -18\sqrt{15}.$$

Example 3.20

Simplify $5\sqrt{18} \cdot 3\sqrt{8}$.

Solution

$$\begin{aligned}5\sqrt{18} \cdot 3\sqrt{8} &= 15\sqrt{2} \cdot 6\sqrt{2} \\ &= 15\sqrt{2} \cdot 6\sqrt{2} \\ &= 180.\end{aligned}$$

Example 3.21

Simplify $\sqrt{\frac{33}{5}} \cdot \sqrt{\frac{15}{11}}$.

Solution

$$\begin{aligned}
\sqrt{\frac{33}{5}} \cdot \sqrt{\frac{15}{11}} &= \sqrt{\frac{33}{5} \cdot \frac{15}{11}} \\
&= \sqrt{3 \cdot 3} \\
&= 3.
\end{aligned}$$

Example 3.22

Simplify $\sqrt{\frac{14}{15}} \cdot \sqrt{\frac{33}{2}} \cdot \sqrt{\frac{10}{7}}$.

Solution

$$\begin{aligned}
\sqrt{\frac{14}{15}} \cdot \sqrt{\frac{33}{2}} \cdot \sqrt{\frac{10}{7}} &= \sqrt{\frac{14}{15} \cdot \frac{33}{2} \cdot \frac{10}{7}} \\
&= \sqrt{22}.
\end{aligned}$$

Example 3.23

Simplify $\sqrt{44}\sqrt{99}$.

Solution

$$\begin{aligned}
\sqrt{44}\sqrt{99} &= 2\sqrt{11} \cdot 3\sqrt{11} \\
&= 6\sqrt{11}\sqrt{11} \\
&= 6\sqrt{11 \cdot 11} \\
&= 6 \cdot 11 \\
&= 66.
\end{aligned}$$

Exercise 3.4

Simplify.

1. $\sqrt{5}\sqrt{5}$
2. $-\sqrt{5}\cdot\sqrt{10}$
3. $\sqrt{50}\cdot\sqrt{32}$
4. $\sqrt{6}\,\sqrt{3}\,\sqrt{8}$
5. $2\sqrt{6}\cdot 3\sqrt{6}$
6. $3\sqrt{12}\cdot 3\sqrt{6}$
7. $2\sqrt{30}\cdot 3\sqrt{15}$
8. $\sqrt{50}\,(-2\sqrt{15})$
9. $\sqrt{10}\sqrt{35}$
10. $\sqrt{\frac{50}{21}}\cdot\sqrt{\frac{14}{15}}$
11. $\sqrt{\frac{15}{77}}\cdot\sqrt{\frac{66}{5}}$
12. $\sqrt{3}\sqrt{-4}$
13. $2\sqrt{245}\sqrt{50}$
14. $\sqrt{175}\sqrt{75}$
15. $\sqrt{30}\sqrt{30}$
16. $2\sqrt{30}\sqrt{45}$
17. $2\sqrt{100}\sqrt{7}$
18. $\sqrt{\frac{26}{7}}\sqrt{\frac{28}{13}}$
19. $\sqrt{\frac{18}{7}}\sqrt{\frac{49}{2}}$
20. $\frac{\sqrt{121}}{11}\cdot\frac{\sqrt{49}}{7}$

3.3.3. Division

Theorem 3.4 (Division rule)

Let a and b be positive numbers. Then $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$.

Example 3.24

Simplify $\sqrt{\dfrac{16}{25}}$.

Solution.

$$\sqrt{\frac{16}{25}} = \frac{\sqrt{16}}{\sqrt{25}} = \frac{4}{5}.$$

Example 3.25

Divide $\sqrt{26}$ by $\sqrt{13}$.

Solution.

$$\frac{\sqrt{26}}{\sqrt{13}} = \sqrt{\frac{\cancel{26}^{2}}{\cancel{13}^{1}}} = \sqrt{2}.$$

An alternative working of Example 3.25 is the following.

Example 3.26

Divide $\sqrt{26}$ by $\sqrt{13}$.

Solution.

$$\frac{\sqrt{26}}{\sqrt{13}} = \frac{\sqrt{13}\sqrt{2}}{\sqrt{13}} = \sqrt{2}.$$

Example 3.27

Simplify $\sqrt{\dfrac{7}{9}}$.

Solution.

$$\sqrt{\frac{7}{9}} = \frac{\sqrt{7}}{\sqrt{9}} = \frac{\sqrt{7}}{3}.$$

3.3.4. Rationalize denominator

A fraction whose denominator is an irrational square root can always be rewritten with a rational denominator. This is called "rationalizing the denominator". This is illustrated by Examples 3.28 to 3.33 on pages 58–59.

Example 3.28

Simplify $\frac{\sqrt{5}}{\sqrt{7}}$.

Solution.

$$\begin{aligned}\frac{\sqrt{5}}{\sqrt{7}} &= \frac{\sqrt{5}}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} \\ &= \frac{\sqrt{35}}{7}.\end{aligned}$$

Example 3.29

Simplify $\frac{1}{\sqrt{2}}$.

Solution.

$$\begin{aligned}\frac{1}{\sqrt{2}} &= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{\sqrt{2}}{2}.\end{aligned}$$

Example 3.30

Simplify $\frac{3}{\sqrt{3}}$.

Solution.

$$\begin{aligned}\frac{3}{\sqrt{3}} &= \frac{3}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} \\ &= \frac{3\sqrt{3}}{3} \\ &= \sqrt{3}.\end{aligned}$$

Example 3.31

Simplify $\frac{7\sqrt{5}}{\sqrt{3}}$.

Solution.

$$\begin{aligned}\frac{7\sqrt{5}}{\sqrt{3}} &= \left(\frac{7\sqrt{5}}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{\sqrt{3}}\right)\\ &= \frac{7\sqrt{15}}{3}.\end{aligned}$$

Example 3.32

Simplify $\frac{11\sqrt{7}}{\sqrt{11}}$.

Solution.

$$\frac{11\sqrt{7}}{\sqrt{11}} = \left(\frac{11\sqrt{7}}{\sqrt{11}}\right)\left(\frac{\sqrt{11}}{\sqrt{11}}\right) = \sqrt{77}.$$

Example 3.33

Simplify $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{5}}$.

Solution.

$$\begin{aligned}\frac{\sqrt{3}+\sqrt{2}}{\sqrt{5}} &= \left(\frac{\sqrt{3}+\sqrt{2}}{\sqrt{5}}\right)\left(\frac{\sqrt{5}}{\sqrt{5}}\right)\\ &= \frac{\sqrt{5}(\sqrt{3}+\sqrt{2})}{\sqrt{5}\sqrt{5}}\\ &= \frac{\sqrt{15}+\sqrt{10}}{5}.\end{aligned}$$

3.3.5. Simplified

The "radicand" is the expression whose square root is taken. $\sqrt{\text{radicand}}$.

An expression is simplified when

(1) the radicand includes no prime factor more than once,
(2) the radicand contains no fractions, and
(3) the denominator contains no square root.

Example 3.34

Divide $\sqrt{6}$ by $\sqrt{5}$.

Solution

$$\sqrt{6} \div \sqrt{5} = \frac{\sqrt{6}}{\sqrt{5}}$$

Not finished, because not yet simplified. Rationalize the denominator.

$$= \left(\frac{\sqrt{6}}{\sqrt{5}}\right)\left(\frac{\sqrt{5}}{\sqrt{5}}\right)$$
$$= \frac{\sqrt{30}}{5}.$$

Example 3.35

$5\sqrt{12} \div 2\sqrt{7}$.

Solution

$$\begin{aligned} 5\sqrt{12} \div 2\sqrt{7} &= \frac{5\sqrt{12}}{2\sqrt{7}} \\ &= \frac{10\sqrt{3}}{2\sqrt{7}} \\ &= \frac{5\sqrt{21}}{7}. \end{aligned}$$

Example 3.36

Simplify $\sqrt{\frac{11}{8}}$.

Solution

$$\begin{aligned} \sqrt{\frac{11}{8}} &= \frac{\sqrt{11}}{\sqrt{8}} \\ &= \frac{\sqrt{11}}{2\sqrt{2}} \\ &= \frac{\sqrt{22}}{4}. \end{aligned}$$

Example 3.37

Simplify $\frac{2}{\sqrt{2}}$.

Solution

$$\begin{aligned} \frac{2}{\sqrt{2}} &= \frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \\ &= \sqrt{2}. \end{aligned}$$

An alternative working would be

$$\frac{2}{\sqrt{2}} = \frac{\sqrt{2} \cdot \sqrt{2}}{\sqrt{2}} = \sqrt{2}.$$

Exercise 3.5

[Part 1] *Rationalize the denominator. See Examples 3.28 - 3.30.*

1. $\frac{\sqrt{2}}{\sqrt{5}}$
2. $\frac{\sqrt{3}}{\sqrt{2}}$
3. $\frac{1}{\sqrt{7}}$
4. $\frac{3}{\sqrt{7}\sqrt{5}}$
5. $\frac{\sqrt{12}}{\sqrt{5}}$
6. $\frac{\sqrt{12}}{\sqrt{3}}$
7. $\frac{\sqrt{12}}{\sqrt{2}}$
8. $\frac{\sqrt{12}}{2\sqrt{3}}$
9. $\frac{\sqrt{5}}{3\sqrt{7}}$
10. $\frac{\sqrt{6}}{2\sqrt{3}}$
11. $\frac{5\sqrt{24}}{3\sqrt{8}}$
12. $\frac{3\sqrt{2}}{\sqrt{-2}}$

[Part 2] *Divide.*

1. $1 \div \sqrt{2}$
2. $\sqrt{18} \div \sqrt{3}$
3. $2\sqrt{3} \div \sqrt{6}$
4. $\sqrt{42} \div \sqrt{14}$
5. $\sqrt{10} \div \sqrt{5}$
6. $\sqrt{96} \div 2\sqrt{3}$

3.3.6. Potpourri

Example 3.38

Perform the computation $5\sqrt{18}+2\sqrt{50}-\sqrt{72}$.

Solution

$$5\sqrt{18}+2\sqrt{50}-\sqrt{72}=15\sqrt{2}+10\sqrt{2}-6\sqrt{2}=19\sqrt{2}.$$

Example 3.39

Simplify $2\sqrt{2}\cdot(2\sqrt{75}-\sqrt{12})$.

Solution

$$2\sqrt{2}\cdot(2\sqrt{75}-\sqrt{12})=2\sqrt{2}\cdot(10\sqrt{3}-2\sqrt{3})=2\sqrt{2}\cdot(8\sqrt{3})=16\sqrt{6}.$$

Example 3.40

Simplify $\dfrac{\sqrt{2}+\sqrt{5}}{\sqrt{3}}$.

Solution

$$\begin{aligned}\frac{\sqrt{2}+\sqrt{5}}{\sqrt{3}} &= \left(\frac{\sqrt{2}+\sqrt{5}}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{\sqrt{3}}\right)\\ &= \frac{\sqrt{3}(\sqrt{2}+\sqrt{5})}{3}\\ &= \frac{\sqrt{6}+\sqrt{15}}{3}.\end{aligned}$$

Example 3.41

Simplify $\dfrac{\sqrt{2}+\sqrt{3}}{\sqrt{3}}+\dfrac{2\sqrt{2}+\sqrt{3}}{\sqrt{2}}$.

Solution

$$\begin{aligned}\frac{\sqrt{2}+\sqrt{3}}{\sqrt{3}}+\frac{2\sqrt{2}+\sqrt{3}}{\sqrt{2}} &= \frac{\sqrt{4}+\sqrt{6}}{\sqrt{6}}+\frac{2\sqrt{6}+\sqrt{9}}{\sqrt{6}}\\ &= \frac{\sqrt{4}+\sqrt{6}+2\sqrt{6}+\sqrt{9}}{\sqrt{6}}\\ &= \frac{5+3\sqrt{6}}{\sqrt{6}}\\ &= \frac{5\sqrt{6}+18}{6}.\end{aligned}$$

Exercise 3.6

Perform the indicated computations. Simplify answers.

1. $3\sqrt{54}-\sqrt{24}$
2. $3\sqrt{20}-2\sqrt{5}$
3. $-\sqrt{3}+3\sqrt{27}$
4. $\sqrt{96}\div 2\sqrt{3}\cdot\sqrt{2}$
5. $\sqrt{54}-3\sqrt{24}\div 4\sqrt{6}$
6. $3\sqrt{21}\times\sqrt{7}\div 9\sqrt{3}$
7. $3\sqrt{20}\div 2\sqrt{5}-\sqrt{45}$
8. $\sqrt{75}-\sqrt{27}+\sqrt{36}$
9. $\sqrt{112}+\sqrt{28}-\sqrt{7}$
10. $\frac{\sqrt{6}}{\sqrt{3}}-\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{2}}$
11. $\frac{2\sqrt{6}}{\sqrt{3}}-\sqrt{128}+\sqrt{\frac{1}{2}}$
12. $\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{5}}$
13. $\frac{\sqrt{2}+\sqrt{7}}{\sqrt{2}}-\frac{\sqrt{21}}{\sqrt{6}}$
14. $\dfrac{\frac{\sqrt{2}+\sqrt{7}}{\sqrt{2}}-\frac{\sqrt{6}+\sqrt{3}}{\sqrt{6}}}{\sqrt{2}}$
15. $3\sqrt{\frac{1}{2}}+2\sqrt{\frac{1}{8}}$
16. $\frac{4\sqrt{5}}{3}+\frac{2}{\sqrt{5}}$
17. $\frac{\sqrt{8}}{3}-\frac{3\sqrt{2}}{3}+\frac{3}{\sqrt{8}}$
18. $3\sqrt{3}+\frac{2}{\sqrt{3}}+\sqrt{\frac{1}{3}}$
19. $\sqrt{\frac{3}{7}}-\frac{2\sqrt{21}}{3}+\frac{1}{\sqrt{21}}$
20. $\frac{12-\sqrt{3}}{2\sqrt{6}}$

Answer the following.

21. Explain why, in the proof of Theorem 3.1 on page 49, Inequality 3.2 follows from Inequality 3.1.
22. Show that $(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})=a-b$ [Hint. Distribution.]
23. Prove that if a and b are each positive, $\sqrt{a}>\sqrt{b}$ implies $a>b$. [Hint. Use the result of problem #22].

Find the value of each of the following if $\sqrt{3.2}=1.789$ *and* $\sqrt{32}=5.657$.

24. $\sqrt{320}$
25. $\sqrt{3200}$
26. $\sqrt{32000}$
27. $\sqrt{0.32}$
28. $\sqrt{0.032}$

3.3.7. Approximation of square roots

We proved that $\sqrt{2}$ is not a rational number. The exact value of $\sqrt{2}$ is $\sqrt{2}$. No rational number, whether written as a common fraction, for example $23/16$, or as a decimal, for example, 1.414, equals $\sqrt{2}$.

It is easy to show that a proposed rational square root of 2 is not exactly equal to $\sqrt{2}$. Just square the proposed number and compare the result to 2.

Example 3.42

Show that neither 1.414 nor $\frac{23}{16}$ is equal to $\sqrt{2}$.

Solution.

$$1.414 \neq \sqrt{2}, \text{ because } (1.414)(1.414) = 1.999396 \neq 2.$$
$$\frac{23}{16} \neq \sqrt{2}, \text{ because } \frac{23}{16} \cdot \frac{23}{16} = \frac{529}{256} \neq 2. \qquad \blacksquare$$

We can derive a formula that will produce a rational number approximately equal to the square root of a number. Suppose we wish to know a rational number approximately equal to $\sqrt{N}$ for $N > 0$. Choose a rational number $a > 0$ that is close to $\sqrt{N}$.

If $a < \sqrt{N}$, then

$$a < \sqrt{N} \tag{3.3}$$
$$1 < \frac{\sqrt{N}}{a}$$
$$\sqrt{N} < \sqrt{N} \cdot \frac{\sqrt{N}}{a}$$
$$\sqrt{N} < \frac{N}{a}. \tag{3.4}$$

Together, inequalities 3.3 and 3.4 imply

$$a < \sqrt{N} < \frac{N}{a}.$$

We may not know the exact value of $\sqrt{N}$. But we know where it lives. We have it trapped between a and $\frac{N}{a}$. See Figure 3.2 on the next page.

We need to select some number in the interval $\left(a, \frac{N}{a}\right)$ as our approximation of $\sqrt{N}$. The midpoint of the interval is our choice, partly because it is an easy number to calculate. The midpoint is the average of the values at the interval's endpoints. Formula 3.5 is our approximation of $\sqrt{N}$.

$$\sqrt{N} \approx \frac{1}{2}\left(a + \frac{N}{a}\right). \tag{3.5}$$

"≈" means "is approximately".

Figure 3.2 graphically shows this strategy. The coordinate of the point P is $\frac{1}{2}\left(a + \frac{N}{a}\right)$.

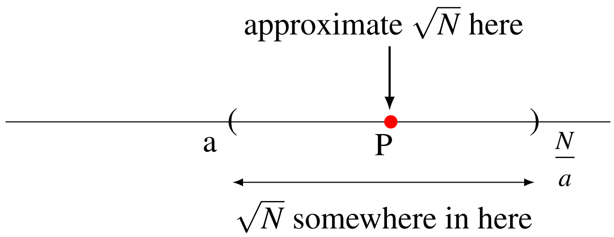


FIGURE 3.2. Approximation of $\sqrt{N}$

Example 3.43

Find a rational approximation of $\sqrt{5}$ accurate to 3 decimal digits.

Solution

Use

$$\sqrt{N} \approx \frac{1}{2}\left(a + \frac{N}{a}\right).$$

Select a number for a by thinking of two consecutive square numbers one less than 5 and the other greater than 5.

$$4 < 5 < 9$$

So,

$$\sqrt{4} < \sqrt{5} < \sqrt{9}$$
$$2 < \sqrt{5} < 3.$$

Since 5 is closer to 4 than to 9, use $a = 2$.

Then

$$\sqrt{N} \approx \frac{1}{2}\left(2 + \frac{5}{2}\right)$$
$$= \frac{1}{2}\left(\frac{9}{2}\right)$$
$$= \frac{9}{4} = 2.25.$$

So, $\sqrt{5} \approx 2.25$. We next use this number 2.25 as a to obtain a new (and closer) approximation of $\sqrt{5}$.

$$\begin{aligned}\sqrt{N} &\approx \frac{1}{2}\left(2.25 + \frac{5}{2.25}\right)\\ &\approx \frac{1}{2}(2.25 + 2.2222)\\ &= 2.2361.\end{aligned}$$

We use 4 decimal digits for intermediate computations, because we seek accuracy to the third digit.

The approximation will be accurate to 3 decimal digits when successive approximations leave the third decimal digit unchanged. We continue with successive approximations until that occurs.

$$\begin{aligned}\sqrt{N} &\approx \frac{1}{2}\left(2.2361 + \frac{5}{2.2361}\right)\\ &= \frac{1}{2}(2.2361 + 2.2360)\\ &= 2.2361.\end{aligned}$$

There is now no change in the third decimal digit. So we report that accurate to 3 decimal digits

$$\sqrt{5} \approx 2.236.$$

How close were the three approximations?

First approximation: $a = 2, \quad 5 - (2.25)(2.25) = -0.0625.$

Second approximation: $a = 2.25, \quad 5 - (2.236)(2.236) = 0.0003.$

Exercise 3.7

1. Use Formula 3.5 on the previous page to approximate each square root with 3 decimal digits accuracy. Please *do* use a calculator for the computations. Then, if your calculator has a square root key, compare your approximation with the calculator's approximation.

   a) $\sqrt{2}$

   b) $\sqrt{6}$

   c) $\sqrt{8}$

   d) $\sqrt{10}$

3.4. nth roots

Since $2 \cdot 2 \cdot 2 = 8$, 2 is the third root of 8. We write the third root of 8 as $\sqrt[3]{8}$.
Since $2 \cdot 2 \cdot 2 \cdot 2 = 16$, 2 is the fourth root of 16. We write the fourth root of 16 as $\sqrt[4]{16}$. Several examples follow.

$$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32 \quad \text{so,} \quad 2 = \sqrt[5]{32}.$$
$$3 \cdot 3 \cdot 3 = 27 \quad \text{so,} \quad 3 = \sqrt[3]{27}.$$
$$3 \cdot 3 \cdot 3 \cdot 3 = 81 \quad \text{so,} \quad 3 = \sqrt[4]{81}.$$
$$7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 = 16807 \quad \text{so,} \quad 7 = \sqrt[5]{16807}.$$

There is a little more to consider. We know that the $\sqrt{a}$ is undefined if $a < 0$. This is so for all even roots. For example, $\sqrt[4]{-16}$ is not a real number. On the other hand, odd roots of negative numbers cause no trouble. For example, $\sqrt[3]{-8} = -2$, because $(-2)(-2)(-2) = 8$.

Definition 3.3 captures the gist of these examples.

Definition 3.3 (nth roots)
For any number a and n a positive integer, $b = \sqrt[n]{a}$ if $b = \underbrace{a \cdot a \cdot a \cdot a \cdot \ \cdots \ \cdot a}_{\text{n factors of } a.}$,
provided that when n is even, a is not negative. ■

All the ideas you acquired working with square roots, extend to nth roots. The examples that follow, should seem completely reasonable to you.

3.4.1. nth simplify

Example 3.44

Simplify the following.

(1) $\sqrt[3]{125}$	(5) $\sqrt[3]{40}$
(2) $\sqrt{81}$	(6) $\sqrt[4]{48}$
(3) $\sqrt[7]{128}$	(7) $\sqrt[3]{72}$
(4) $\sqrt[3]{54}$	(8) $\sqrt[3]{216}$

Solution.

(1) $\sqrt[3]{125} = \sqrt[3]{5 \cdot 5 \cdot 5} = 5$

(2) $\sqrt{81} = 9, \quad \sqrt{81}$ means $\sqrt[2]{81}$.

(3) $\sqrt[7]{2} = \sqrt[7]{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2} = 2$

(4) $\sqrt[3]{54} = \sqrt[3]{2 \cdot 3 \cdot 3 \cdot 3} = 3\sqrt[3]{2}$

(5) $\sqrt[3]{40} = \sqrt[3]{2 \cdot 2 \cdot 2 \cdot 5} = 2\sqrt[3]{5}$

(6) $\sqrt[4]{48} = \sqrt[4]{2 \cdot 2 \cdot 2 \cdot 2 \cdot 3} = 2\sqrt[4]{3}$

(7) $\sqrt[3]{72} = \sqrt[3]{2 \cdot 2 \cdot 2 \cdot 3 \cdot 3} = 2\sqrt[3]{9}$

(8) $\sqrt[3]{216} = \sqrt[3]{2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3} = 2 \cdot 3 = 6$ ■

Writing multiple factors is instructive. Once the point that

$$a = \underbrace{\sqrt[n]{a \cdot a \cdot a \cdot a \cdot \ \cdots \ \cdot a}}_{\text{n factors of } a.}$$

has been made, we can save some effort by writing

$$\underbrace{a \cdot a \cdot a \cdot a \cdot \ \cdots \ \cdot a}_{\text{n factors of } a.} \text{ as } a^n.$$

The result is wonderfully efficient notation,

$$\sqrt[n]{a^n} = a.$$

Providing that $a \geq 0$ whenever n is even. Example 3.45 reworks Example 3.44 on the preceding page taking advantage of this notation.

Remark 3.1
Later, we will prove that

$$\sqrt[n]{a^n} = (\sqrt[n]{a})^n = a.$$

Example 3.45

Simplify the following. This is Example 3.44 on the previous page all over again.

(1) $\sqrt[3]{125}$

(2) $\sqrt[4]{81}$

(3) $\sqrt[7]{128}$

(4) $\sqrt[3]{54}$

(5) $\sqrt[3]{40}$

(6) $\sqrt[4]{48}$

(7) $\sqrt[3]{72}$ (8) $\sqrt[3]{216}$

Solution.

(1) $\sqrt[3]{125} = \sqrt[3]{5^3} = 5$

(2) $\sqrt[4]{81} = \sqrt[4]{3^4} = 3$

(3) $\sqrt[7]{2} = \sqrt[7]{2^7} = 2$

(4) $\sqrt[3]{54} = \sqrt[3]{2 \cdot 3^3} = 3\sqrt[3]{2}$

(5) $\sqrt[3]{40} = \sqrt[3]{2^3 \cdot 5} = 2\sqrt[3]{5}$

(6) $\sqrt[4]{48} = \sqrt[4]{2^4 \cdot 3} = 2\sqrt[4]{3}$

(7) $\sqrt[3]{72} = \sqrt[3]{2^3 \cdot 3^2} = 2\sqrt[3]{9}$

(8) $\sqrt[3]{216} = \sqrt[3]{2^3 \cdot 3^3} = 2 \cdot 3 = 6$

Exercise 3.8

Simplify.

1. $4\sqrt[3]{375}$
2. $3\sqrt[4]{48}$
3. $5\sqrt[3]{256}$
4. $-4\sqrt[4]{32}$
5. $-5\sqrt[3]{81}$
6. $5\sqrt[4]{162}$
7. $4\sqrt[3]{500}$
8. $8\sqrt[4]{486}$
9. $-4\sqrt[3]{-192}$
10. $3\sqrt[5]{224}$

3.4.2. nth roots addition (subtraction)

If you expect to read "like terms combine, unlike terms do not," then you are ahead of the game. But, what counts as like terms in the context of nth roots?

Definition 3.4 (Like terms for nth roots)
Providing that $a \geq 0$ when m is even and $b \geq 0$ when n is even, $\sqrt[m]{a}$ and $\sqrt[n]{b}$ are **like terms** if $a = b$ and $m = n$.

Like: $\sqrt[3]{5}$ and $6\sqrt[3]{5}$ combine: $\sqrt[3]{5} + 6\sqrt[3]{5} = 7\sqrt[3]{5}$.
Unlike: $\sqrt[3]{6}$ and $6\sqrt[3]{5}$. Do not combine.
Unlike: $\sqrt[3]{5}$ and $6\sqrt[4]{5}$. Do not combine.

3.4.3. nth roots multiplication

The rule is $\sqrt[m]{a} \cdot \sqrt[m]{b} = \sqrt[m]{ab}$, provided a and b are positive when m is even. The roots must be identical. $\sqrt[m]{a} \cdot \sqrt[n]{b}$ does not simplify unless $m = n$.

Example 3.46

Simplify the following.

(1) $\sqrt[3]{7} \cdot \sqrt[3]{4} = \sqrt[3]{28}$.
(2) $\sqrt[5]{2} \cdot \sqrt[5]{7} = \sqrt[5]{14}$.
(3) $\sqrt[3]{7} \cdot \sqrt[4]{7}$. Does not simplify
(4) $\sqrt[3]{-6} \cdot \sqrt[3]{7} = \sqrt[3]{-42}$.
(5) $\sqrt[4]{-3} \cdot \sqrt[3]{7}$. Is undefined.

3.4.4. nth roots division

The rule is $\dfrac{\sqrt[m]{a}}{\sqrt[m]{b}} = \sqrt[m]{\dfrac{a}{b}}$, provided a and b are positive when m is even and $b \neq 0$ when m is odd. The roots must be identical.

Example 3.47

Divide

(1) $\dfrac{\sqrt[3]{5}}{\sqrt[3]{7}} = \sqrt[3]{\dfrac{5}{7}}$.
(2) $\dfrac{\sqrt[4]{7}}{\sqrt[4]{2}} = \sqrt[4]{\dfrac{7}{2}}$.
(3) $\dfrac{\sqrt[3]{5}}{\sqrt[4]{7}}$. Cannot rewrite.
(4) $\dfrac{\sqrt[4]{32}}{\sqrt[4]{2}} = \sqrt[4]{\dfrac{32}{2}} = \sqrt[4]{16} = 2$.

3.4.5. nth roots rationalize denominator

Denominators must be rational in simplified expressions.

Example 3.48

Simplify $\frac{\sqrt[3]{5}}{\sqrt[3]{7}}$.

Solution

How many factors of $\sqrt[3]{7}$ must we multiply $\sqrt[3]{7}$ by to produce 7? Two, because $\sqrt[3]{7} \cdot \sqrt[3]{7} \cdot \sqrt[3]{7} = 7$.

$$\begin{aligned}\frac{\sqrt[3]{5}}{\sqrt[3]{7}} &= \left(\frac{\sqrt[3]{5}}{\sqrt[3]{7}}\right)\left(\frac{\sqrt[3]{7}\sqrt[3]{7}}{\sqrt[3]{7}\sqrt[3]{7}}\right) \\ &= \frac{\sqrt[3]{5}\sqrt[3]{7}\sqrt[3]{7}}{\sqrt[3]{7}\sqrt[3]{7}\sqrt[3]{7}} \\ &= \frac{\sqrt[3]{245}}{7}.\end{aligned}$$

Example 3.49

Simplify $\frac{4\sqrt[3]{5}}{\sqrt[3]{4}}$.

Solution

How many factors of $\sqrt[3]{4}$ must we multiply $\sqrt[3]{4}$ by to produce 4? Two, because $\sqrt[3]{4} \cdot \sqrt[3]{4} \cdot \sqrt[3]{4} = 4$.

$$\begin{aligned}\frac{4\sqrt[3]{5}}{\sqrt[3]{4}} &= \left(\frac{4\sqrt[3]{5}}{\sqrt[3]{4}}\right)\left(\frac{\sqrt[3]{4}\sqrt[3]{4}}{\sqrt[3]{4}\sqrt[3]{4}}\right) \\ &= \frac{4\sqrt[3]{5}\sqrt[3]{4}\sqrt[3]{4}}{\sqrt[3]{4}\sqrt[3]{4}\sqrt[3]{4}} \\ &= \sqrt[3]{80} \\ &= 2\sqrt[3]{10}.\end{aligned}$$

Exercise 3.9

Simplify.

1. $-2\sqrt{5}-2\sqrt{3}-2\sqrt{5}$
2. $-\sqrt{2}+2\sqrt{2}-2\sqrt{2}$
3. $2\sqrt[3]{4}+2\sqrt[3]{-4}-2\sqrt[3]{4}$
4. $-2\sqrt{3}+2\sqrt{2}-\sqrt{2}$
5. $2\sqrt[4]{2}+2\sqrt[4]{3}-2\sqrt[4]{3}$
6. $2\sqrt{5}-\sqrt{5}+2\sqrt{3}$
7. $-\sqrt[4]{3}-\sqrt[4]{3}-2\sqrt[4]{4}$
8. $2\sqrt{2}-2\sqrt{2}+2\sqrt{3}$
9. $2\sqrt{3}-\sqrt{3}-\sqrt{3}$
10. $2\sqrt{3}+2\sqrt{3}+2\sqrt{3}$
11. $2\sqrt[3]{32}+2\sqrt[3]{3}-2\sqrt[3]{32}$
12. $2\sqrt[3]{4}-\sqrt[3]{16}-\sqrt[3]{4}$
13. $2\sqrt[4]{2}+2\sqrt[4]{64}-2\sqrt[4]{2}$
14. $-2\sqrt[3]{2}-\sqrt[3]{16}-\sqrt[4]{2}$
15. $-\sqrt[4]{64}-2\sqrt[4]{4}+2\sqrt[4]{2}$
16. $-2\sqrt[3]{24}-\sqrt[3]{24}-\sqrt[3]{24}$
17. $-2\sqrt[3]{4}+2\sqrt[3]{4}-\sqrt[3]{24}$
18. $-\sqrt[3]{4}+2\sqrt[4]{4}-2\sqrt[4]{3}$
19. $-2\sqrt[3]{16}-2\sqrt[3]{16}+2\sqrt[3]{4}$
20. $-\sqrt[3]{2}-2\sqrt[3]{16}+2\sqrt[3]{16}$
21. $-\sqrt[3]{16}-2\sqrt[3]{3}+2\sqrt[3]{2}$
22. $2\sqrt[3]{32}+2\sqrt[4]{32}-\sqrt[3]{-32}$
23. $-\sqrt[3]{4}-2\sqrt[3]{32}-2\sqrt[4]{48}$
24. $-\sqrt[4]{32}-\sqrt[4]{64}+2\sqrt[4]{32}$
25. $-2\sqrt[3]{32}+2\sqrt[3]{32}-2\sqrt[3]{16}$
26. $2\sqrt[2]{3}-\sqrt[3]{16}-\sqrt[3]{-32}$
27. $-2\sqrt[3]{3}-\sqrt[3]{4}+\sqrt[3]{24}$
28. $2\sqrt[3]{2}-\sqrt[3]{16}+2\sqrt[3]{16}$
29. $-\sqrt[4]{3}-2\sqrt[4]{64}-2\sqrt[4]{48}$
30. $-2\sqrt[4]{4}-2\sqrt[4]{2}-\sqrt[4]{4}$
31. $\sqrt[3]{72}\cdot\sqrt[3]{-18}$
32. $-6\sqrt[4]{9}\cdot\sqrt[4]{108}$
33. $\sqrt[4]{64}\cdot\sqrt[4]{80}$
34. $-3\sqrt[6]{96}\cdot 4\sqrt[6]{12}$
35. $\sqrt[4]{20}\cdot\sqrt[4]{4}$
36. $3\sqrt[4]{12}\left(-5\sqrt[4]{108}\right)$
37. $\sqrt[4]{135}\cdot\sqrt[4]{6}$
38. $\sqrt[3]{12}\cdot\sqrt[3]{24}$
39. $\sqrt[4]{36}\cdot\sqrt[4]{72}$
40. $-5\sqrt[4]{48}\cdot\sqrt[4]{16}$

Simplify.

41. $\dfrac{\sqrt[4]{3}}{4\sqrt[4]{4}}$
42. $\dfrac{2\sqrt[3]{3}}{\sqrt[3]{-16}}$
43. $\dfrac{2}{4\sqrt[3]{-9}}$
44. $\dfrac{\sqrt[3]{8}}{4\sqrt[3]{12}}$
45. $\dfrac{\sqrt[3]{3}}{2\sqrt[3]{-4}}$
46. $\dfrac{\sqrt[4]{3}}{\sqrt[4]{27}}$
47. $\dfrac{\sqrt[6]{8}}{3\sqrt[6]{16}}$
48. $\dfrac{\sqrt[4]{3}}{\sqrt[4]{9}}$
49. $\dfrac{2\sqrt[4]{4}}{\sqrt[4]{8}}$
50. $\dfrac{4\sqrt[3]{3}}{4\sqrt[3]{4}}$

3.5. Prime numbers revisited

The task of determining whether or not a number is prime is easier, if we use some facts about numbers and their square roots.

Example 3.50

Is the number 163 a prime number?

Solution

The number 163 is not divisible by $2, 3, 5, 7, 11$ or 13. Conclusion: 163 is a prime number. ■

You might be ready to accuse the author of shoddy workmanship for stopping at 13. There are prime numbers between 13 and 163. Maybe one of them is a factor of 163.

There is an idea needed for the author to vindicate himself. We consider it now.

Theorem 3.5
If N has no prime factor p such that $0 < p \leq \sqrt{N}$, then N has no prime factor q such that $\sqrt{N} \leq q < N$. In this case N must be a prime number.

Proof. Suppose there is a prime factor of N, call it q, with $\sqrt{N} \leq q < N$. Then N is a composite number and as such must have another prime factor. Call that prime factor p. Now $p \leq \sqrt{N}$ otherwise $p \cdot q > \sqrt{N} \cdot \sqrt{N} = N$. Equivalently, if N has no prime factor $p \leq \sqrt{N}$ then N has no prime factor $\sqrt{N} \leq q < N$. ■

Theorem 3.5 means that if we test the primes in their natural order $2, 3, 5, 7, \cdots$ and reach a prime greater than $\sqrt{N}$ without having found a prime factor of N, then we may as well quit. We will find no prime factor of N greater than $\sqrt{N}$. In such a case, N is a prime number.

As for Example 3.50, $163 < 169$, so $\sqrt{163} < \sqrt{169} = 13$. According to Theorem 3.5, 163 is a prime number, because none of $\{2, 3, 5, 7, 11, 13\}$ are factors of 163.

Example 3.51

Is the number 223 a prime number?

Solution

Since $15^2 = 225 > 223$, we do not need to test primes larger than 15. Now 223 is not divisible by $2, 3, 5, 7, 11$ or 13 and $17 > 15$, so we are done. The number 223 is prime.

Example 3.52

Is the number 389 a prime number?

Solution

Since $20^2 = 400 > 389$, we do not need to test primes larger than 20. Now 389 is not divisible by $2, 3, 5, 7, 11, 13, 17$ or 19 and $23 > 20$, so we are done. The number 389 is prime.

Exercise 3.10

(a) List the prime numbers you must test to see if the given number is prime.
(b) State whether the number is prime or non-prime.

1. 74
2. 107
3. 109
4. 193
5. 397
6. 657
7. 659
8. 137
9. 157
10. 177
11. 231
12. 233

Chapter 4

Exponents

In this chapter, we begin with positive integer exponents. As the story unfolds, we extend exponents to include 0, negative integers, and the rational numbers.

4.1. Positive integer exponent

Positive integer exponents provide a convenient way of writing several factors of a number. For example, we can write 5 factors of 2 as 2^5 instead of $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$. Definition 4.1 commemorates this idea.

Definition 4.1 (a^n)
When a is any number and n is a positive integer, a^n means n factors of a. The number a is called the base and the number n is called the exponent. ■

We can think of a^n like this

$$a^n = \underbrace{a \cdot a \cdot a \cdot a \cdot a \cdot \ldots \cdot a}_{\text{n factors of a}}.$$

$$\text{base}^{\text{exponent}}$$

FIGURE 4.1. Naming parts of a^n.

Other words used for "exponent" are "power" and sometimes "index". The symbol a^n is often pronounced "a to the n" or "a to the nth power". Regardless of how you pronounce 2^5, it is well to think "5 factors of 2". By tradition, instead of writing a number to the 1st power, we just write the number. For example, 5^1 is written 5.

Sections 4.1.1 and 4.1.2 explain computations with exponents.

4.1.1. Products

Theorem 4.1 (Product rule for exponents)
When m and n are positive integers and a is any number,

$$a^m \cdot a^n = a^{m+n}.$$

■

Theorem 4.1 should seem reasonable when we think of the meanings of a^m and a^n.

$$a^m \cdot a^n = \underbrace{\underbrace{a \cdot a \cdot a \cdot a \cdot a \cdot \ldots \cdot a}_{m \text{ factors of } a} \cdot \underbrace{a \cdot a \cdot a \cdot a \cdot a \cdot \ldots \cdot a}_{n \text{ factors of } a}}_{m+n \text{ factors of } a}.$$

We write $m+n$ factors of a as a^{m+n}. Therefore, $a^m \cdot a^n = a^{m+n}$.

Be careful to note that Theorem 4.1, the product rule, requires one base. To see why the product rule does not apply when there are different bases, consider the meanings of a^m and b^n.

$$a^m \cdot b^n = \underbrace{\underbrace{a \cdot a \cdot a \cdot a \cdot a \cdot \ldots \cdot a}_{m \text{ factors of } a} \cdot \underbrace{b \cdot b \cdot b \cdot b \cdot b \cdot \ldots \cdot b}_{n \text{ factors of } b}}_{m+n \text{ factors of ????????}}.$$

We cannot say of what there are $m+n$ factors. It is the old "can't add apples and oranges" routine.

Example 4.1

Using the product rule.

(1) $3^5 \cdot 3^6 = 3^{5+6} = 3^{11}$

(2) $4^3 \cdot 4^2 = 4^{3+2} = 4^5$

(3) $7^5 \cdot 7 = 7^{5+1} = 7^6$

(4) $11^2 \cdot 11^3 = 11^{2+3} = 11^5$

(5) $2^3 \cdot 2^5 \cdot 3^2 \cdot 3^4 = 2^8 \cdot 3^6$

(6) $2 \cdot 3^2 \cdot 3^5 = 2 \cdot 3^7$

Exercise 4.1

Simplify. Most of these involve the product rule. Some, like question #2, just require the definition of a^n. A few require both the definition and product rule.

1. -3^2
2. $(-3)^2$
3. -3^3
4. $(-3)^3$
5. $2^3 \cdot 2^2$
6. $7 \cdot 7^2$
7. $2^3 \cdot 2^2 \cdot 2^2$
8. $2^2 \cdot 3^2$
9. $x^2 \cdot x^7$
10. $y^3 \cdot y^2$
11. $x^3 \cdot x^7 \cdot x^5$
12. $x \cdot x^2$
13. $x^3 \cdot y^5$
14. $10^3 \cdot 10^2 \cdot 3^4$
15. $2 \cdot 2^2 \cdot 2^3$
16. $x^2 \cdot y^5 \cdot 2^3 \cdot x^5$
17. $(-2)^3 \cdot 2^2$
18. $-2^4 \cdot 2^2$
19. $(-3)^3 \cdot 2^2 \cdot 3$
20. $(-1)^7$
21. $(-1)^8$
22. $(2x)^3 \cdot x^4$
23. $(2x)^2 \cdot (2y)^3$
24. $2x^5 \cdot (2x)^2$
25. $(2y)^2 \cdot (-2y)^3 \cdot 3$
26. $\left(\frac{2}{3}\right)^3$

4.1.2. Powers

Keeping in mind the meaning of a^n, we expect that $(2x)^5 = 32x^5$, because

$$(2x)^5 = (2x)(2x)(2x)(2x)(2x) = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot x \cdot x \cdot x \cdot x \cdot x$$

is the product of 5 factors of 2 and 5 factors of x. We could use the product rule.

$$\begin{aligned}(2x)^5 &= (2x)(2x)(2x)(2x)(2x)\\ &= 2^1 \cdot 2^1 \cdot 2^1 \cdot 2^1 \cdot 2^1 \cdot x^1 \cdot x^1 \cdot x^1 \cdot x^1 \cdot x^1\\ &= 2^{(1+1+1+1+1)}x^{(1+1+1+1+1)}\\ &= 2^5x^5\end{aligned}$$

But we have something different in mind. These computations can be performed using the "power rule for exponents".

Theorem 4.2 (Power rule for exponents)
Let all real numbers a and b and m and n positive integers.

Part a: Power of a power. $(a^m)^n = a^{mn}$.

Part b: Power of a product. $(ab)^n = a^n b^n$.

Part c: Power of a quotient. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$, providing $b \neq 0$.

Proof. Let a and b be any real numbers and m and n positive integers.
Part a. $(a^m)^n$ means n factors of a^m and a^m means m factors of a.

$$(a^m)^n = \underbrace{\underbrace{a \cdot a \cdot a \ldots \cdot a}_{m \text{ factors of } a} \cdot \underbrace{a \cdot a \cdot a \ldots \cdot a}_{m \text{ factors of } a} \cdot \underbrace{a \cdot a \cdot a \ldots \cdot a}_{m \text{ factors of } a} \cdot \ldots \cdot \underbrace{a \cdot a \cdot a \ldots \cdot a}_{m \text{ factors of } a}}_{n \text{ groups of } m \text{ factors of } a = nm \text{ factors of } a.}$$

We write nm factors of a as a^{mn}. Therefore $(a^m)^n = a^{mn}$.

Part b. $(ab)^n$ means n factors of ab.

$$(ab)^n = \underbrace{ab \cdot ab \cdot ab \cdot \ldots \cdot ab}_{n \text{ factors of } ab} = \underbrace{a \cdot a \cdot a \ldots \cdot a}_{a^n} \cdot \underbrace{b \cdot b \cdot b \ldots \cdot b}_{b^n} = a^n b^n.$$

The proof of Part (c) is left as an exercise. ■

The proof of Theorem 4.2 used the meaning of a^n. Write out the theorem and its proof on your own a few times. You will understand it quite well.

Example 4.2

Simplify $(2x)^5$.

Solution

$(2x)^5 = 2^5x^5$ according to the power rule for exponents.

Example 4.3

Simplify.

(1) $\left(2^3\right)^2$

(2) $\left(a^4\right)^3$

(3) $(ab)^5$

(4) $\left(a^2b^4\right)^5$

(5) $\left(\dfrac{a}{b}\right)^3$

(6) $\left(\dfrac{b^2}{a^7}\right)^4$

Solution

(1) $2^{2\cdot 3} = 2^6$

(2) $a^{3\cdot 4} = a^{12}$

(3) a^5b^5

(4) $\left(a^2\right)^5\left(b^4\right)^5 = a^{10}b^{20}$

(5) $\dfrac{a^3}{b^3}$

(6) $\dfrac{\left(b^2\right)^4}{\left(a^7\right)^4} = \dfrac{b^8}{a^{28}}$

Example 4.4

Neither of these expressions can be further simplified, because the bases do not match.

(1) $3^5 \cdot 2^6$. (2) $2^4 \cdot 3^4$. ■

Sometimes people confuse the expressions $-a^n$ and $(-a)^n$. If you are careful to identify what the n-factors are *of*, you can avoid this confusion.

$$-a^n \text{ means } -\underbrace{a\cdot a\cdot a\cdot a\cdot a\cdot\ldots\cdot a}_{n\text{ factors of } a}$$

$$(-a)^n \text{ means } \underbrace{(-a)\cdot(-a)\cdot(-a)\cdot(-a)\cdot(-a)\cdot\ldots\cdot(-a)}_{n\text{ factors of } (-a)}$$

Examples 4.5 to 4.6 on the following page illustrate this idea.

Example 4.5 (1) $-2^4 = -2 \cdot 2 \cdot 2 \cdot 2 = -16$.

(2) $(-2)^4 = (-2) \cdot (-2) \cdot (-2) \cdot (-2) = 16$.

Example 4.6

Simplify $(-3x)^2(-3x^5)$.

Solution

$$\begin{aligned}(-3x)^2(-3x^5) &= (-3x) \cdot (-3x) \cdot -3(x \cdot x \cdot x \cdot x \cdot x) \\ &= (-3)(-3)(-3) \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \\ &= -27x^7.\end{aligned}$$ ■

The point of Example 4.6 is to make the distinction of $(-a)^n$ and $(-a^n)$ obvious. In practice, the working of Example 4.6 would be like the following.

$$\begin{aligned}(-3x)^2(-3x^5) &= (-3)^2 \cdot x^2 \cdot (-3x^5), \quad \text{power rule} \\ &= (-3)^2 \cdot x^2 \cdot (-3)x^5 \\ &= (-3)^3 x^7, \quad \text{product rule} \\ &= -27x^7.\end{aligned}$$

4.2. Zero exponent

We want to extend the set of numbers that can serve as exponents. The trouble is, we have no clue what, if anything, a^0 names. Definition 4.1 on page 75 limits itself to positive integer exponents. It says nothing about a^0. If we naively extend Definition 4.1 to include 0, we would say that a^0 means "0 factors of a". But what does that mean? We need a different approach.

The product rule, Theorem 4.1 on page 76, captures a characteristic of exponents that seems essential to a thing's being an exponent. Intuitively, exponents should behave just as the product rule says they do. If we admit 0 to the tribe of numbers that may serve as exponents, then we expect 0 to obey the product rule, too.

Suppose that a^0 does behave according to the product rule. Then,

$$a^n \cdot a^0 = a^{n+0} = a^n.$$

Conclusion: If a^0 follows the product rule, then a^0 names the number 1.

Definition 4.2 (Zero exponent)

For any number $a \neq 0$, $a^0 = 1$. ■

Exercise 4.2

Simplify.

1. $\left(m^4\right)^3$
2. $\left(b^2\right)^0$
3. $\left(3x^2\right)^2$
4. $\left(4n^3\right)^2$
5. $\left(2v^3\right)^2$
6. $\left(x^4y^3\right)^3$
7. $\left(2u^2v^3\right)^4$
8. $\left(a^4b^2\right)^3$
9. $\left(4vu^4\right)^2$
10. $\left(3y^4\right)^2$
11. $\left(-x^0\right)^4$
12. $\left(2k^2\right)^2$
13. $\left(2a^3\right)^3$
14. $\left(-2x^3\right)^4$
15. $(-3n)^4$
16. $-\left(2p^2\right)^4$
17. $\left(-3xy^3\right)^3$
18. $\left(3x^0y^4\right)^4$
19. $\left(-y^2\right)^3$
20. $\left(-4vu^4\right)^2$
21. $\left(-x^4\right)^2$
22. $\left(-4x^3y^2\right)^3$
23. $\left(\frac{2}{3}\right)^4$
24. $\left(\frac{a}{2}\right)^5$
25. $\left(\frac{a^3}{b}\right)^2$
26. $\left(\frac{u^3}{v^5}\right)^3$
27. $\left(\frac{-r^3}{t^2}\right)^2$
28. $\left(\frac{(-u)^5}{v^6}\right)^3$
29. $\left(\frac{(-1)}{v^2}\right)^{11}$
30. $\left(\frac{-x^2}{(-y)^3}\right)^3$
31. $\left(\frac{-9(-x)^5y^{21}}{\left(-7^2x^5\right)^9}\right)^0$

4.3. Negative integer exponent

No previous definition tells us the meaning of a^n where $n < 0$. Let's proceed as we did for a^0. Suppose $a \neq 0$ and $n > 0$ so that $-n$ is negative. Further suppose negative exponents behave as we know exponents *should* behave. Then,

$$a^n \cdot a^{-n} = a^{n+(-n)} = a^0 = 1.$$

This means that a^{-n} is the multiplicative inverse of a^n. That is,

$$a^{-n} = \frac{1}{a^n}.$$

Definition 4.3 (Negative integer exponent)
For any number $a \neq 0$, and n a positive integer, $a^{-n} = \frac{1}{a^n}$. ■

Now that negative integer exponents have been defined, we extend the power rule to include them. Showing the product rule holds with exponent 0 is an exercise question. Thus $a^m b^n = a^{m+n}$ for any integers m and n.

It is sometimes convenient to rewrite a^n as $\frac{1}{a^{-n}}$ when n is a positive integer. The next theorem says we can do so.

Theorem 4.3
For $a \neq 0$ and a positive integer n, $a^n = \frac{1}{a^{-n}}$.

Proof. Let a be any nonzero real number and n a positive integer.

$$\begin{aligned}\frac{1}{a^{-n}} &= \frac{1}{\frac{1}{a^n}} \\ &= 1 \cdot \frac{a^n}{1} \\ &= a^n.\end{aligned}$$

■

4.3.1. Exponent facts for integer exponents

The exponent rules extended to include all integer exponents. For all integers m and n and all real numbers a and b,

Zero exponent. $a^0 = 1, \quad a \neq 0.$

Negative exponent. $a^{-n} = \dfrac{1}{a^n}, \quad a \neq 0.$

Product rule. $a^m \cdot a^n = a^{m+n}.$

Quotient rule. $\dfrac{a^m}{a^n} = a^{m-n}, \quad a \neq 0.$

Power rules

$(a^m)^n = a^{mn}.$

$(ab)^n = a^n b^n.$

$\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}, \quad b \neq 0.$

Example 4.7

Simplify each expression. Write all answers with positive exponents. All letters represent nonzero real numbers.

(1) x^{-7}

(2) $\dfrac{1}{a^{-2}}$

(3) $\dfrac{a^{-3}}{a^2}$

(4) $\dfrac{x^2}{x^{-5}}$

(5) $\dfrac{x^{-3}}{y^{-2}}$

(6) $\dfrac{10^{-3}}{10^{-5}}$

Solution

(1) $\dfrac{1}{x^7}$

(2) a^2

(3) $a^{-3}a^{-2} = a^{-5} = \dfrac{1}{a^5}$

(4) $x^{2-(-5)} = a^7$

(5) $\dfrac{y^2}{x^3}$

(6) $10^{-3-(-5)} = 10^2 = 100$

Exercise 4.3

Simplify each expression. Write all answers with positive exponents. All letters represent nonzero real numbers.

1. $\frac{n^{-3}}{n^{-1}}$
2. $\frac{x^{-4}}{3x}$
3. $\frac{3b^{-4}}{4b^{2}}$
4. $\frac{2p^{-1}}{p^{2}}$
5. $\frac{r^{-1}}{r^{3}}$
6. $\frac{k}{2k^{2}}$
7. $\frac{b^{-1}}{4b}$
8. $\frac{2n^{0}}{2n^{2}}$
9. $\frac{4x^{2}}{x^{-4}}$
10. $\frac{3p^{-3}}{p^{0}}$
11. $\frac{3x^{-4}}{5x^{5}}$
12. $\frac{x^{0}}{3x^{-1}}$
13. $\frac{4v^{-5}}{v^{4}}$
14. $\frac{4p^{-5}}{4p^{-1}}$
15. $\frac{4n^{-1}}{5n^{-3}}$
16. $\frac{3x^{-4}}{x^{-4}}$
17. $\frac{5n}{5n^{-5}}$
18. $\frac{3x^{5}}{4x^{2}}$
19. $\frac{3n^{-2}}{4n^{-5}}$
20. $\frac{5m^{-2}}{4m^{4}}$
21. $\frac{4r^{-2}}{4r^{2}}$
22. $\frac{x^{-2}}{4x^{3}}$
23. $\frac{3x^{-2}}{5x^{3}}$
24. $\frac{2b^{-1}}{b}$
25. $\frac{5k^{0}}{5k}$
26. $\frac{k^{-4}}{2k^{4}}$

Answer the following.

27. Show that the product rule holds for exponents of 0.

Several rules are used to simplify a complicated expression.

Example 4.8

Simplify $\left(\dfrac{-x^{-2}\cdot x^{-5}}{-x^{-3}\left(x^{9}\right)^{-3}}\right)^{10}$.

Solution

$$\begin{aligned}\left(\frac{-x^{-2}\cdot x^{-5}}{-x^{-3}\left(x^{9}\right)^{-3}}\right)^{10} &= \left(\frac{-x^{-7}}{-x^{-3}\left(x^{9}\right)^{-3}}\right)^{10}, \quad \text{Product rule.}\\ &= \left(\frac{-x^{-7}}{-x^{-3}x^{-27}}\right)^{10}, \quad \text{Power rule.}\\ &= \left(\frac{x^{-7}}{x^{-30}}\right)^{10}, \quad \text{Product rule.}\\ &= \left(x^{23}\right)^{10}, \quad \text{Product rule.}\\ &= x^{230}, \quad \text{Power rule.}\end{aligned}$$

Example 4.9

Simplify $\left(-\dfrac{x^{7}\cdot x^{-3}}{\left(-x^{5}\left(-x^{2}\right)\right)^{7}}\right)^{2}$.

Solution

$$\begin{aligned}\left(-\frac{x^{7}\cdot x^{-3}}{\left(-x^{5}\left(-x^{2}\right)\right)^{7}}\right)^{2} &= \left(-\frac{x^{4}}{\left(-x^{5}\left(-x^{2}\right)\right)^{7}}\right)^{2}, \quad \text{Product rule.}\\ &= \left(-\frac{x^{4}}{\left(x^{7}\right)^{7}}\right)^{2}, \quad \text{Product rule.}\\ &= \left(-\frac{x^{4}}{x^{49}}\right)^{2}, \quad \text{Power rule.}\\ &= \frac{x^{8}}{x^{98}}, \quad \text{Power rule.}\end{aligned}$$

Exercise 4.4

Simplify each expression. Write all answers with positive exponents. All letters represent nonzero real numbers.

1. $\dfrac{x^{-8}x^{8}}{\left(x^{7}\right)^{5}x^{-5}}$

2. $\dfrac{\left(r^{5}\right)^{3}r^{8}}{r^{7}r^{0}}$

3. $\dfrac{\left(x^{3}\left(x^{-6}\right)^{5}\right)^{-3}}{\left(x^{2}\right)^{-2}}$

4. $\dfrac{\left(p^{-6}\right)^{-5}}{p^{4}p^{2}}$

5. $\dfrac{b^{-1}}{\left(b^{7}\right)^{7}b^{5}}$

6. $\dfrac{\left(n^{7}\right)^{-3}n^{5}}{n^{0}}$

7. $\left(\dfrac{\left(r^{0}\left(r^{3}\right)^{0}\right)^{2}}{r^{0}}\right)^{-2}$

8. $\dfrac{m^{0}m^{5}\left(m^{-1}\right)^{7}}{m^{3}}$

9. $\left(\dfrac{x^{-7}x^{0}}{x^{8}}\right)^{-5}$

10. $\left(\dfrac{v^{-4}v}{\left(v^{5}\right)^{-5}}\right)^{-2}$

11. $\left(\dfrac{k^{2}k^{-3}}{\left(k^{2}\right)^{3}}\right)^{-5}$

12. $\dfrac{\left(a^{-3}\right)^{-7}}{-a^{3}\left(-a^{-6}\right)}$

13. $\dfrac{v^{6}v^{-2}}{\left(v^{3}\right)^{5}}$

14. $\dfrac{\left(-x^{2}\right)^{6}\left(-x^{8}\right)}{x^{-3}}$

15. $-\dfrac{x^{8}}{x^{-1}\left(x^{2}\right)^{2}}$

16. $-\dfrac{p^{8}}{\left(-p^{2}\left(-p^{0}\right)\right)^{3}}$

17. $-\dfrac{k^{-1}}{\left(-k^{-2}\right)^{8}k^{3}}$

18. $-\dfrac{x^{0}}{\left(x^{3}x^{2}\right)^{5}}$

19. $\left(\dfrac{-n^{-2}}{-n^{-5}\left(n^{6}\right)^{8}}\right)^{8}$

20. $\dfrac{\left(-n^{0}n^{0}\right)^{2}}{n^{5}}$

4.4. Rational exponents

We have extended the numbers we use as exponents as far as the integers. We now ask what, if anything, a rational exponent means. For example what could $x^{1/2}$ mean?

Again, we insist that any number in the role of exponent must behave as we are sure exponents should behave. In particular, such a number must follow the product rule, $a^m \cdot a^n = a^{m+n}$.

Suppose $x^{1/2}$ follows the product rule. Then

$$x^{1/2} \cdot x^{1/2} = x^{1/2+1/2} = x^1 = x.$$

Apparently, $x^{1/2}$ behaves just like $\sqrt{x}$.

What about $x^{1/3}$?

$$x^{1/3} \cdot x^{1/3} \cdot x^{1/3} = x^{1/3+1/3+1/3} = x^1 = x.$$

$x^{1/3}$ behaves like the third root of x, $\sqrt[3]{x}$.

Exponents ought to follow the power rule, too. By the power rule, $a^{1/n}$ raised to the power n results in a. That is,

$$(a^{1/n})^n = a^{\frac{1}{n} \cdot n} = a.$$

The nth root of a raised to the power n also results in a. That is,

$$\left(\sqrt[n]{a}\right)^n = a.$$

Apparently, $a^{1/n}$ and $\sqrt[n]{a}$ are names for the nth root of a.

$$a^{1/n} = \sqrt[n]{a}.$$

We need to be a little careful defining $a^{1/n}$ as the nth root of a, because when $a < 0$ and n is an even number, $\sqrt[n]{a}$ does not exist.

Definition 4.4 (Rational exponent)
Let n represent a positive integer.

$$a^{1/n} \text{ is the nth root of } a, \text{ if } \begin{cases} n \text{ is an odd number and } a \text{ is any real number.} \\ n \text{ is even and } a \text{ is any nonnegative real number.} \end{cases}$$ ■

Several examples clarify Definition 4.4.

Example 4.10

Simplify with rational exponents.

(1) $3^{1/4} = \sqrt[4]{3}$.

(2) $\left(\frac{2}{3}\right)^{1/3} = \sqrt[3]{\frac{2}{3}}$.

(3) $(-8)^{1/3} = \sqrt[3]{-8}$.

(4) $(-8)^{1/2}$ is undefined.

(5) $(16)^{1/2} = \sqrt{16} = 4$.

We have defined $a^{1/n}$. Now rational exponents in general are within our reach. All that remains is to say what $a^{m/n}$ means. The power rule provides two paths to our goal.

(1) $a^{m/n} = a^{(1/n)m} = (a^{1/n})^m$.
(2) $a^{m/n} = a^{m(1/n)} = (a^m)^{1/n}$.

As the scarecrow says in *The Wizard of Oz*, "Some people go both ways." That is what we will do.

Definition 4.5
Let m and n represent positive integers and let m/n be fully simplified. Then,

$$a^{m/n} = (a^{1/n})^m = (a^m)^{1/n},$$

provided $a^{1/n}$ is defined. ∎

Since $a^{1/n} = \sqrt[n]{a}$,

$$a^{m/n} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}. \tag{4.1}$$

Equation 4.1 provides a choice. Example 4.11 is about how to make that choice.

Example 4.11

Evaluate $32^{6/5}$.

Solution

Equation 4.1 allows the choice of using

$$\sqrt[5]{32^6} \quad \text{or} \quad \left(\sqrt[5]{32}\right)^6.$$

Both computations are shown.

The Bad: $\sqrt[5]{32^6} = \sqrt[5]{1073741824} = 64$.

The Good: $\left(\sqrt[5]{32}\right)^6 = 2^6 = 64$. ∎

Negative integer exponents have been defined. Definition 4.6 does the honors for negative rational exponents.

Definition 4.6
Let m and n represent positive integers and let m/n be fully simplified. Then,

$$a^{-m/n} = \frac{1}{a^{m/n}}, \qquad \text{provided } a \neq 0.$$

Example 4.12

Write in radical form.

(1) $8^{7/3} = (\sqrt[3]{8})^7$ or $(8)^{7/3} = \sqrt[3]{8^7}$

(2) $(5x)^{2/9} = (\sqrt[9]{5x})^2$ or $(5x)^{2/9} = (\sqrt[9]{(5x)^2})$

Write in exponential form.

(3) $(\sqrt[3]{8})^5 = 8^{5/3}$

(4) $(\sqrt[8]{(4u)^5} = (4u)^{5/8}$

(5) $\frac{1}{(\sqrt[3]{5})^7} = 5^{-7/3}$

Exercise 4.5

[Part 1] *Write each expression in radical form.*

1. $7^{1/3}$
2. $5^{2/3}$
3. $10^{3/2}$
4. $7^{4/3}$
5. $6^{1/2}$
6. $5^{4/3}$
7. $(6b)^{1/2}$
8. $x^{7/6}$
9. $(6k)^{4/3}$
10. $(5x)^{3/2}$
11. $m^{4/3}$
12. $(6r)^{3/2}$

[Part 2] *Write each expression in exponential form.*

1. $(\sqrt[6]{10})^7$
2. $(\sqrt[3]{3})^2$
3. $\sqrt[3]{7^2}$
4. $\sqrt[4]{3^7}$
5. $(\sqrt[3]{4})^5$
6. $\sqrt[2]{6^5}$
7. $(\sqrt[3]{2r})^2$
8. $\sqrt[2]{(3m)^5}$
9. $\sqrt[4]{(10x)^3}$
10. $(\sqrt[4]{3n})^7$
11. $\sqrt[4]{(3k)^5}$
12. $(\sqrt[4]{2p})^3$

[Part 3] *Write each expression in radical form.*

1. $(10b)^{-7/6}$
2. $(4n)^{-5/3}$
3. $(x)^{3/2}$
4. $(2x)^{-1/6}$
5. $(4x)^{-1/3}$
6. $(4v)^{-2/3}$

[Part 4] *Write each expression in exponential form.*

1. $\dfrac{1}{(\sqrt[3]{3r})^2}$
2. $\dfrac{1}{(\sqrt{x})^5}$
3. $\dfrac{1}{(\sqrt[3]{7r})^4}$
4. $\dfrac{1}{(\sqrt[6]{n})^5}$
5. $\dfrac{1}{\sqrt[3]{6v^2}}$
6. $(\sqrt[4]{5n})^5$

4.5. Exponents and radicals

In Section 4.4 we found that we could go back and forth writing an expression using exponents or radicals. This is handy, because sometimes it is more convenient to work in one form than in the other.

Example 4.13

Simplify $\sqrt[3]{8x^3y^6} \div \sqrt{xy^{-2}}$.

Solution

$$\begin{aligned}\sqrt[3]{8x^3y^6} \div \sqrt{xy^{-2}} &= (8x^3y^6)^{1/3} \div (xy^{-2})^{1/2}\\ &= 2xy^2 \div x^{1/2}y^{-1}\\ &= 2xy^2 \cdot x^{-1/2}y\\ &= 2x^{1/2}y^3.\end{aligned}$$

If for some reason a radical form is preferred

$$2x^{1/2}y^3. = 2y^3\sqrt{x}.$$

Example 4.14

Simplify $\dfrac{2}{\sqrt{x}} \cdot \dfrac{1}{\frac{1}{\sqrt{x}} + \sqrt{x}}$.

Solution

$$\begin{aligned}\frac{2}{\sqrt{x}} \cdot \frac{1}{\frac{1}{\sqrt{x}} + \sqrt{x}} &= \frac{2}{\sqrt{x}\left(\frac{1}{\sqrt{x}} + \sqrt{x}\right)}\\ &= \frac{2}{1+x}\end{aligned}$$

Example 4.15

Simplify $2x^{-1/2} \cdot (x^{-1/2} + x^{1/2})^{-1}$.

Solution

$$\begin{aligned}2x^{-1/2} \cdot (x^{-1/2} + x^{1/2})^{-1} &= 2x^{-1/2} \cdot \frac{1}{x^{-1/2} + x^{1/2}}\\ &= \frac{2x^{-1/2}}{x^{-1/2} + x^{1/2}}\\ &= \frac{2x^{-1/2}}{x^{-1/2} + x^{1/2}} \cdot \frac{x^{1/2}}{x^{1/2}}\\ &= \frac{2x^0}{x^0 + x^1} = \frac{2}{1+x}.\end{aligned}$$

■

Examples 4.14 and 4.15 come to the same conclusion, because

$$\frac{2}{\sqrt{x}} \cdot \frac{1}{\frac{1}{\sqrt{x}} + \sqrt{x}} = 2x^{-1/2} \cdot (x^{-1/2} + x^{1/2})^{-1}.$$

4.6. Summary

For convenience, a complete summary is provided below of all the rules of exponents you have learned. Those for integer exponents from page 83 are reproduced here. Remember that the set of rational numbers includes the integers. So any rule the holds for rational numbers holds for integers.

Definitions. Let m and n represent integers, r represent a rational number, and a represent a real number,

(1) For $n > 0$, a^n means n factors of a.

(2) $a^1 = a$.

(3) 0^0 is undefined.

(4) For $a \neq 0$, $a^0 = 1$.

(5) For $a^r \neq 0$, $a^{-r} = \dfrac{1}{a^r}$.

(6) For $n \neq 0, a^m \neq 0^0$, and $a^m \geq 0$ when n is even, $a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$.

Sometimes 0^0 defined to be 1, in order to simplify certain formulae. In this book, 0^0 stays undefined.

The disclaimers such as "$a^m \geq 0$ when n is even" make the facts appear complicated. The disclaimers are all about avoiding undefined expressions. We can take care of the disclaimers all at once by saying "*providing all expressions are defined*". But then it is up to you to spot possible division by zero or possible even roots of negative numbers.

Rules. Let a and b represent any real numbers. Let r and s be any rational numbers. Then, providing the expression is defined,

(1) Product. $a^r \cdot a^s = a^{r+s}$.

(2) Quotient. $\dfrac{a^r}{a^s} = a^{r-s}$.

(3) Power of a power. $(a^r)^s = a^{rs}$.

(4) Power of product. $(ab)^r = a^r b^r$.

(5) Power of quotient. $\left(\dfrac{a}{b}\right)^r = \dfrac{a^r}{b^r}$.

Exercise 4.6

Write the numerical value of each expression without exponents or radicals.

1. $9^{1/2}$
2. $25^{-1/2}$
3. $64^{2/3}$
4. $\left(\frac{1}{5}\right)^{-2}$
5. $27^{-1/3}$
6. 3^0
7. $(-4)^{1/2}$
8. $8^{5/3}$
9. $\left(\frac{1}{27}\right)^{-1/3}$
10. $(-2)^{-2}$
11. $100^{3/2}$
12. $125^{-2/3}$
13. $\left(\frac{1}{2}\right)^{-5}$

Simplify. Exponents in answer must be positive. No radicals in answer

14. $x^{10} \cdot x^{-3}$
15. $(x^{33})^{1/3}$
16. $\sqrt[3]{-8}$
17. $x^7 \div x^8$
18. $(x^{-1})^{-1}$
19. $x^{1/5} \cdot x^{3/5} \cdot x^{2/5}$
20. $x^{1/3} \cdot x^{1/4}$
21. $(27x^{-3})^{-1/3}$
22. $(x^5)^5$
23. $x^{1/2} \div x^{-1/2}$
24. $x^{1/2}(x^{1/2} + x^{-1/2})$
25. $(3x^{-1/2})^{-2}$
26. $\sqrt{x^{10}}$
27. $\sqrt[5]{\frac{1}{x^5}}$

4.6.1. Review problems

This section features problems that invite you to orchestrate several ideas to produce a solution.

Exercise 4.7

Answer the following.

1. Show that $(x^2+y^2)^{-1/2}$ is not equal to $x^{-1}+y^{-1}$.
2. Simplify $\dfrac{x^{3/2}-x^{-3/2}}{x^{-3/2}}$.
3. Simplify $2^{-n}\cdot 8^{n-1}\cdot 4^{n+3}\div 16^n$. [*Hint*: express each factor as a power of 2].
4. Simplify $\left(\dfrac{10^{\,n+2}}{100}\right)^{1/n}$.
5. Simplify $2^x\cdot 2$.
6. Show that $2^{x+2}-(2^{x+1}+2^x)=2^x$.
7. Prove Part c of Theorem 4.2 on page 78.
8. Show that the power rule on page 78 works when the exponent is 0.
9. Prove the quotient rule $\dfrac{a^r}{a^s}=a^{r-s}$.
10. How many factors has 1 billion. ($1\text{billion}=10^9$.)

4.6.2. The path we traveled

The first numbers to serve as exponents were the positive integers. We understood the symbol a^n to mean n factors of a. For example, 2^3 means three factors of 2.

As candidates for exponents, we examined in turn zero, negative integers, then rational numbers. We may not have expected that zero, negative integers and rational numbers would serve as exponents. But they do.

4.6.3. Loose end

Let us look once again at a question from Section 2.2 on page 25. Is the the graph of $y = x^{1.01}$ a straight line?

If we can find just two pairs of points on the line $y = x^{1.01}$ such that the slope computed using one pair does not equal the slope using the other pair, we will have shown that $y = x^{1.01}$ is not straight. Select points $P(0, 0^{1.01}), Q(1, 1^{1.01}), R(2, 2^{1.01})$. Then,

$$\text{slope using PQ} = \frac{1^{1.01} - 0^{1.01}}{1 - 0} = 1^{1.01} = 1$$
$$\text{slope using PR} = \frac{2^{1.01} - 1^{1.01}}{2 - 1} = 2^{1.01} - 1$$

If the slopes are equal, then

$$1 = 2^{1.01} - 1$$
$$2 = 2^{1.01} \tag{4.2}$$

This is false. So there is at least one portion of $y = x^{1.01}$ that is not straight. Although Equation 4.2 is false, the difference between $2^{0.01}$ and 1 is very small. $2^{0.01} - 1 \approx 0.007$. This may explain why $y = x^{1.01}$ appears straight even when magnified by a factor of 10.

4.7. Sermonette

The Chapter about the straight line and this chapter about exponents each contain a lot of content compared to what you have had in the past. The content in the straight line chapter included all those different forms of the equation of a straight line. This chapter includes a bucket full of exponent rules. As you advance through mathematics, the topics you meet will include more and more content. Way too much to know from memory. To continue to do well (and to enjoy mathematics) means discovering just how little you need to remember about a topic in order to produce all the content that you will not memorize.

If you want to learn mathematics this way, try this. Starting with the idea that “a straight line does not change direction”, try to reproduce the reasoning in the text that produced all the other ideas and equations. You will get stuck. When you do, go back and have a peek at the book. Then continue until you get stuck again. Repeat peeking at book and continuing. You may have to go through this process many times. Each time, the flow of the discussion will make more sense to you. The flow will eventually seem quite natural and the progression of ideas and equations obvious to you. You will realize you will always succeed in getting the fact you need even if you have to go back to “a straight line does not change direction”. You will not fear “forgetting something” because you will have nothing to remember other than a few obvious basic ideas. You will truly know mathematics.

To practice learning mathematics this way using the chapter on exponents, start with the idea that a^n means n number of factors of a. Try to get all the rest to unfold as in the text.

Chapter 5

Factorization

Let us briefly review some useful vocabulary. Numbers or products connected by "+" or "−" are called *terms*. For example, the expressions $5x+3y-6$ consists of the three terms "$5x$", "$3y$",' and '6". An expression having but one term is called a *monomial*. An expression having two terms is called a *binomial*. And, no surprise, an expression having three terms is called a *trinomial*. In general, monomials, binomials, trinomials, etcetera are called polynomials. A *polynomial* is any number of monomials connected by "+" or "−" . Yes, each term is a monomial. The *degree of a term* is the highest exponent that occurs on a letter in that term. For example, the degree of $3x^5$ is 5. There is a little more to be said about degree, but it will come up in a subsequent course. The *degree of a polynomial* is the degree of the highest degree term in that polynomial. For example, the highest degree term in $x^4+3x^2-10x+12$ is x^4, so the polynomial is degree 4.

We will mainly be concerned with trinomials. The terms of trinomials have names that will be convenient for us to use. If the trinomial is arranged in order of descending degree, then from left to right, the first term is called the *leading term*, the second the *middle term* or *cross term* or *cross product*. The last term is called the *last term*. If the last term is a constant it is often called *the constant term* or just *the constant*.

5.1. Products to sums

You have often found the product of polynomials. Your computations were limited to the product of a monomial and a binomial. For example, $3(x+5) = 3x+15$. We now wish to multiply polynomials having more terms than one and two. You already possess the ideas and skills that will get the job done.

Example 5.1

Find the product $(x+2)(x+3)$.

Solution

Distribute the blob $(x+2)$,

$$(x+2)(x+3) = x(x+2)+3(x+2).$$

Distribute,

$$= x^2+2x+3x+6.$$

Combine like terms,

$$= x^2+5x+6. \qquad \blacksquare$$

Instead of saying "find the product" or "multiply" when working with polynomials, we usually say "expand".

Example 5.2

Expand $(2x+5)(x+7)$.

Solution

$$\begin{aligned}(2x+5)(x+7) &= (2x+5)x+(2x+5)7\\ &= 2x^2+5x+14x+35\\ &= 2x^2+19x+35.\end{aligned}$$

Example 5.3

Expand $(2x+3)(5x+7)$.

Solution

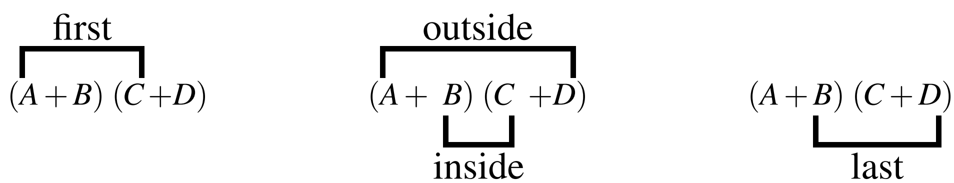
$$\begin{aligned}(2x+3)(5x+7) &= (2x+3)5x+(2x+3)7\\ &= 10x^2+15x+14x+35\\ &= 10x^2+29x+35. \qquad \blacksquare\end{aligned}$$

If you study examples 5.1 to 5.3 on the facing page, you might notice a pattern. To better see the pattern, we replace the first and second terms of one binomial with “A”, “B”, and the terms of the second binomial with “C” and “D”.

$$\begin{aligned}(A+B)(C+D) &= (A+B)C+(A+B)D \\ &= AC+BC+AD+BD. \qquad \blacksquare\end{aligned}$$

We call

AC**:** the product of the first terms,
BC**:** the product of the inside terms,
AD**:** the product of the outside terms,
BD**:** the product of the last terms.



With a little practice, you will apply this pattern mentally, instead of writing out the double distribution.

Example 5.4

Expand $(3x+5)(2x+3)$.

Solution

$$(\underset{A}{3x}+\underset{B}{5})(\underset{C}{2x}+\underset{D}{3}) = \underset{AC}{6x^2}+\underset{BC}{10x}+\underset{AD}{9x}+\underset{BD}{15}$$

combining like terms

$$= 6x^2+19x+15.$$

Example 5.5

Expand $(x-4)(x+5)$.

Solution

$$\begin{aligned}(x-4)(x+5) &= x^2-4x+5x-20 \\ &= x^2+x-20.\end{aligned}$$

Example 5.6

Expand $(x-5)(2x-3)$.

Solution

$$\begin{aligned}(x-5)(2x-3) &= 2x^2 - 10x - 3x + 15 \\ &= 2x^2 - 13x + 15.\end{aligned}$$ ■

Examples 5.6 to 5.8 on pages 99–100 should remind you to keep track of signs.

When there are like terms, they are usually combined mentally as in Examples 5.7 and 5.8.

Example 5.7

Expand $(x+2)(7x+5)$.

Solution

$$(x+2)(7x+5) = 7x^2 + 19x + 10.$$

Example 5.8

Expand $(x-6)(2x+3)$.

Solution

$$(x-6)(2x+3) = 2x^2 - 9x - 18.$$

Exercise 5.1

Expand.

1. $(5x+1)(2x+5)$
2. $(k+7)(k-3)$
3. $(2x+2)(3x+1)$
4. $(7n+6)(8n+4)$
5. $(6x-3)(4x-2)$
6. $(7m+7)(7m+8)$
7. $(2v-4)(7v-6)$
8. $(4x-8)(3x+7)$
9. $(6n-2)(8n+1)$
10. $(8n-6)(2n-7)$
11. $(4n+2)(6n-3)$
12. $(2x+7)(3x+1)$
13. $(3v-8)(6v+2)$
14. $(p+6)(7p-5)$
15. $(7x+2)(6x+5)$
16. $(r+4)(5r+4)$
17. $(5x-1)(6x+6)$
18. $(3r-6)(7r-6)$
19. $(4n+4)(n+4)$
20. $(3x-5)(4x-1)$
21. $(3x-3)(6x+7)$
22. $(2r+4)(6r+4)$
23. $(6n+1)(2n-7)$
24. $(b+4)(3b-3)$
25. $(6n-6)(5n+3)$
26. $(8n+4)(4n+2)$
27. $(7k-6)(k+8)$
28. $(8n+7)(5n-7)$
29. $(3a+6)(3a-4)$
30. $(5x+1)(8x-2)$

5.2. Sums to products

The axiom of distribution enables us to write a product as a sum. For example,

$$(x+2)(x+5) = x^2 + 7x + 10.$$

Distribution also enables us to write a sum as a product. For example,

$$x^2 + 9x + 20 = (x+4)(x+5).$$

While it is usually easy to write a product as a sum, it can be far from obvious how to rewrite a sum as a product. The process of rewriting a sum as a product is called "factoring". We begin factoring the least difficult polynomials. We gradually build up to the "far from obvious" kind. Factoring polynomials is a skill that improves with experience.

Since there is not much by way of practically useful theory, we will rely primarily on examples. The author's goal is to provide the kinds of experiences that will help you to become skillful at factoring in the least time with the least effort. You would be wise to carefully work *every* example along with the book.

5.3. Trinomials whose leading coefficient is 1

The phrase "leading coefficient" denotes the coefficient of the polynomial's highest degree term. For example, the highest degree term of $3x^2 + 2x - 4$ is $3x^2$, so the leading coefficient is 3.

Examples 5.9 to 5.13 on pages 102–104 are quite similar. That is intentional. The first few examples will have annotated solutions. The letters 'A', 'B', 'C', and 'D' that appear in the equation below are only present so we can refer to the various places in the parenthesis.

Example 5.9

Rewrite $x^2 + 5x + 6$ as a product.

Solution

The goal is to fill in each parenthesis so that the product is $x^2 + 5x + 6$. That is,

$$x^2 + 5x + 6 = (A \qquad B)(C \qquad D).$$

There is no doubt about positions A and C. They can only be occupied by 'x'. So,

$$x^2 + 5x + 6 = (x \qquad B)(x \qquad D).$$

The constant 6 is positive, so it is the product of two negatives or two positives. Based on experience expanding, we know the coefficient of the middle term results from combining like terms by adding them. Since this sum is 5, a positive number, the factors of 6 are both positive.

$$x^2+5x+6=(x+B)(x+D).$$

Do we use 1 and 6 or 2 and 3 in positions B and D? The middle term $5x$ is the sum of $2x$ and $3x$. So,

$$x^2+5x+6=(x+2)(x+3).$$

Alway check a factorization as soon as you produce it. The check is as easy as multiplying two polynomials as in Exercise 5.1 on page 101. Check:

$$(x+2)(x+3)=x^2+2x+3x+6=x^2+5x+6. \checkmark$$

Example 5.10

Rewrite x^2-5x+6 as a product.

Solution

The goal is to fill in each parenthesis so that the product is x^2-5x+6. That is,

$$x^2-5x+6=(A \qquad B)(C \qquad D).$$

To obtain x^2, positions A and C must be occupied by 'x'. So,

$$x^2-5x+6=(x \qquad B)(x \qquad D).$$

The constant 6 is positive, so it is the product of two negatives or two positives. Since the coefficient of the middle term is negative, the factors are both negative.

$$x^2-5x+6=(x-B)(x-D).$$

Do we use 1 and 6 or 2 and 3 in positions B and D? The middle term $-5x$ is the sum of $-2x$ and $-3x$. So,

$$x^2-5x+6=(x-2)(x-3).$$

Check:

$$(x-2)(x-3)=x^2-2x-3x+6=x^2-5x+6. \checkmark$$

Example 5.11

Factor x^2+x-6.

Solution

$$x^2+x-6=(A \qquad B)(C \qquad D)$$

$$= (x \quad B)(x \quad D).$$

The constant is -6. So, it must be the product of a negative and a positive number.

$$x^2 + x - 6 = (x + B)(x - D).$$

Since the middle term is positive, the positive number must be more positive than the negative number is negative. Do we use 1 and 6 or 2 and 3 in positions B and D? The middle term is x and $3x - 2x = x$. So,

$$x^2 + x - 6 = (x + 3)(x - 2).$$

Check:

$$(x + 3)(x - 2) = x^2 + 3x - 2x - 6 = x^2 + x - 6. \checkmark$$

Example 5.12

Factor $x^2 - x - 6$.

Solution

$$x^2 - x - 6 = (x \quad B)(x \quad D).$$

The constant is -6. So, it must be the product of a negative and a positive number.

$$x^2 - x - 6 = (x + B)(x - D).$$

Since the middle term is negative, the negative number must be more negative than the positive number is positive. Do we use 1 and 6 or 2 and 3? The middle term is $-x$ and $2x - 3x = -x$. So,

$$x^2 - x - 6 = (x + 2)(x - 3).$$

Check:

$$(x + 2)(x - 3) = x^2 + 2x - 3x - 6 = x^2 - x - 6. \checkmark$$

Example 5.13

Factor $x^2 + 7x + 6$.

Solution

$$x^2 + 7x + 6 = (x \quad B)(x \quad D).$$

The constant is 6. So, positive $\times$ positive *or* negative $\times$ negative. Since middle term is positive,

$$x^2 + 7x + 6 = (x + B)(x + D).$$

Do we use 1 and 6 or 2 and 3? The middle term is $7x$ and $x + 6x = 7x$. So,

$$x^2 + 7x + 6 = (x + 1)(x + 6).$$

Check:

$$(x+1)(x+6) = x^2 + x + 6x + 6 = x^2 + 7x + 6. \checkmark$$

■

Example 5.14

Factor $-x^2 - 2x + 15$.

Solution

The factorization is a little simpler when the leading term does not include the "-" sign.

$$\begin{aligned} -x^2 - 2x + 15 &= -[x^2 + 2x - 15] \\ &= -[(x-3)(x+5)] \\ &= -(x-3)(x+5) \quad (5.1) \\ &= (-x+3)(x+5) \quad (5.2) \end{aligned}$$

Either Equation 5.1 or Equation 5.2 is an acceptable answer. Equation 5.1 would generally be preferred. ■

The examples so far have included a constant term of 6 or -6. Since 6 has few factors, there were not so many possibilities. When the constant has many factors, the trial and error nature of factoring polynomials becomes apparent. Even so, a little thought can sometimes eliminate useless guesses.

Example 5.15

Factor $x^2 + 13x + 12$.

Solution

$x^2 + \mathbf{13}x + 12 =$	$(x+\quad)$	$(x+\quad)$	
	1	12	Yup: $\mathbf{1} + \mathbf{12} = \mathbf{13}$
	2	6	
	3	4	

So, $x^2 + 13x + 12 = (x+1)(x+12)$.

Check: $(x+1)(x+12) = x^2 + x + 12x + 12 = x^2 + 13x + 12. \checkmark$

Example 5.16

Factor $x^2 + 7x + 12$.

Solution

$$\begin{array}{lccl} x^2+\mathbf{7}x+12=(\,x+\quad)\;(\,x+\quad) & & & \\ & 1 & 12 & \\ & 2 & 6 & \\ & 3 & 4 & \text{Yup: } \mathbf{3}+\mathbf{4}=\mathbf{7} \end{array}$$

Check: $(x+3)(x+4)=x^2+3x+4x+12=x^2+7x+12$. ✓

Therefore, $x^2+7x+12=(x+3)(x+4)$.

Example 5.17

Factor $x^2-4x-12$.

Solution

$$\begin{array}{lccl} x^2-\mathbf{4}x-12=(\,x+\quad)\;(\,x-\quad) & & & \\ & 1 & 12 & \\ & 2 & 6 & \text{Yup: } \mathbf{2}-\mathbf{6}=-\mathbf{4} \\ & 3 & 4 & \end{array}$$

Check: $(x+2)(x-6)=x^2+2x-6x-12=x^2-4x-12$. ✓

Therefore, $x^2-4x-12=(x+2)(x-6)$. ■

When the teacher worked Example 5.17 on the board, she started out this way . . .

$$\begin{array}{lcc} x^2-\mathbf{4}x-12=(\,x+\quad)\;(\,x-\quad) & & \\ & 12 & 1 \\ & 6 & 2 \\ & 4 & 3 \end{array}$$

But she got no further, because a student protested that the teacher's plan could not succeed. Was the student correct? Yes. The teacher's arrangement cannot produce the negative coefficient on the middle term.

Exercise 5.2

Factor the expression.

1. $x^2-2x-48$
2. p^2+p-90
3. $x^2+15x+50$
4. n^2+8n-9
5. $b^2-8b-20$
6. p^2+4p+3
7. b^2-7b+6
8. a^2+a-42
9. $k^2-3k-28$
10. v^2-5v+6
11. $x^2-2x-80$
12. $x^2+13x+42$
13. $n^2+17n+70$
14. $n^2+16n+60$
15. x^2+x-6
16. x^2+x-20
17. $r^2-13r+30$
18. $x^2+14x+48$
19. $x^2+2x-80$
20. x^2+2x-8
21. $a^2-17a+72$
22. $a^2+5a-24$
23. $r^2+7r+12$
24. $n^2+4n-45$
25. $x^2-8x-20$
26. $x^2+10x+9$
27. $n^2-4n-45$
28. $n^2-14n+45$
29. $n^2-7n+12$
30. p^2-4p-5
31. $v^2-2v-15$
32. $b^2-5b-50$
33. x^2-2x-3
34. b^2+b-20
35. n^2-8n+7
36. k^2+4k-5
37. r^2+5r-6
38. $v^2-5v-24$
39. x^2+8x+7
40. $n^2+6n-27$
41. $n^2-14n+40$
42. $k^2-2k-35$
43. $k^2+7k+10$
44. $m^2-10m+24$
45. $m^2-10m+16$
46. r^2+6r+5
47. x^2-x-90
48. $x^2+11x+24$
49. $m^2-11m+30$
50. $x^2+5x-36$
51. $x^2+13x+42$
52. b^2-2b-8
53. $n^2-9n+14$
54. $x^2-2x-63$
55. v^2-5v-6
56. $x^2-2x-35$
57. $p^2+4p-12$
58. $n^2-5n-36$
59. $a^2+4a-60$
60. $m^2-11m+24$

5.4. Trinomials: leading coefficient a prime number.

When the leading coefficient is not 1, the polynomial is harder to factor because there are more possibilities to consider. But, so long as the leading coefficient is a prime number, there are not too many more possibilities.

Example 5.18

Factor $2x^2+5x+3$.

Solution

There is only one way to produce $2x^2$. Also, both factors of the constant 2 must be positive. So,

$$\begin{array}{rcc} 2x^2+5x+3 = (\,2x+ &) & (\,x+ \quad) \\ & 1 & 3 \\ & 3 & 1 \end{array}$$

The first pair $(1,3)$ results in

$$\begin{aligned}(2x+1)(x+3) &= 2x^2+x+6x+3 \\ &= 2x^2+7x+3.\end{aligned}$$

But,

$$2x^2+7x+3 \neq 2x^2+5x+3.$$ ■

It was not necessary to completely expand $(2x+1)(x+3)$. If we had checked the middle term first, we would have known this trial fails. *As a rule, always check the middle term first.*

The second pair $(3,1)$ produces the correct middle term $3x+2x=5x$ ✓. So we check the whole expansion.

$$(2x+3)(x+1) = 2x^2+3x+2x+3 = 2x^2+5x+3 \checkmark.$$

Example 5.19

Factor $5x^2-17x-12$.

Solution

There is only one way to produce $5x^2$. Also, one factor of 12 must be negative, the other positive and the negative factor must be more negative than the positive factor is positive.

$$5x^2 - 17x - 12 = (\ 5x + \quad)\ \ (\ x - \quad)$$

1	12
2	6
3	4

The 1st pair $(1,12)$ yields a middle term of $5(-12)+1 = -59 \neq -17$. NO!
The 2nd pair $(2,6)$ yields a middle term of $5(-6)+2 = -28 \neq -17$. NO!
The 3rd pair $(3,4)$ produces a middle term of $5(-4)+3 = -17$. ✓
Therefore, $5x^2 - 17x - 12 = (5x+3)(x-4)$.

Example 5.20

Factor $7x^2 + 110x - 32$.

Solution

There is only one way to produce $7x^2$. Also, one factor of 32 must be negative, the other positive and the positive factor must be more positive than the negative factor is negative.

$$7x^2 + 110x - 32 = (\ 7x - \quad)\ \ (\ x + \quad)$$

1	32
2	16
4	8

The 1st pair $(1,32)$ yields a middle term of $7(32)-1 = 223 \neq 110$. NO!
The 2nd pair $(2,16)$ yields a middle term of $7(16)-2 = 110$. ✓
Therefore, $7x^2 + 110x - 32 = (7x-2)(x+16)$.

Exercise 5.3

Factor each expression.

1. $3m^2-25m+28$
2. $3x^2-14x+15$
3. $5n^2-19n-30$
4. $7b^2+76b+60$
5. $7x^2+23x-20$
6. $3p^2+8p-35$
7. $5n^2+17n-12$
8. $7r^2+58r+63$
9. $7m^2-46m-80$
10. $2n^2-3n-2$
11. $3x^2+22x+40$
12. $7x^2+34x-5$
13. $5n^2+43n-18$
14. $7a^2-29a-30$
15. $3v^2-14v-24$
16. $3r^2-19r-72$
17. $7k^2+64k-60$
18. $3x^2+20x-100$
19. $3k^2-29k+40$
20. $3x^2-10x-8$
21. $5n^2-3n-8$
22. $5p^2-32p+12$
23. $7v^2+23v+18$
24. $3x^2+14x-80$
25. $5b^2+48b-20$
26. $5n^2-38n+48$
27. $5m^2+36m+7$
28. $3v^2-4v+1$
29. $5p^2+37p-72$
30. $3m^2+16m+20$
31. $7m^2-20m-32$
32. $7r^2-3r-10$
33. $2r^2-17r+21$
34. $7x^2-62x-80$
35. $3b^2+23b+14$
36. $5b^2+37b+42$
37. $3v^2+v-30$
38. $7p^2+25p+12$
39. $7n^2+61n+40$
40. $5x^2-43x-18$
41. $7n^2+52n-32$
42. $7m^2-44m+45$
43. $3n^2-19n+28$
44. $2p^2-13p+18$
45. $7a^2-59a+70$
46. $2v^2-v-6$
47. $2x^2+9x+7$
48. $2x^2+7x-30$
49. $2x^2-15x-27$
50. $5x^2-52x+63$
51. $7x^2+43x-42$
52. $2n^2+13n+6$
53. $2r^2-21r+40$
54. $2x^2-x-36$
55. $3v^2+20v+25$
56. $7n^2-46n-21$
57. $5v^2+54v+40$
58. $7n^2+52n+21$
59. $7v^2+17v-12$
60. $7p^2-10p+3$

5.5. Annoying cases

When the leading coefficient is neither 1 nor prime, the polynomial may be annoying to factor by trial and error, because of the large number of possibilities. Consider factoring $12x^2 - 11x - 36$. The divisors of 12 are $1, 2, 3, 4, 6, 12$. The divisors or 36 are $1, 2, 3, 4, 6, 9, 12, 18, 36$. This is probably no one's idea of a good time.

We will tackle $12x^2 - 11x - 36$ on page 118 after learning a strategy that takes some of the misery out of factoring an expression like $12x^2 - 11x - 36$. Before we can discuss that strategy, we need the technique called "factoring by grouping" – the topic of the next section.

5.6. Factoring by grouping

Recall that we expanded $(ax+b)(cx+d)$ by just using distribution twice and then combining like terms.

$$(ax+b)(cx+d) = cx(ax+b) + d(ax+b) \tag{5.3}$$

$$= acx^2 + bcx + adx + bd. \tag{5.4}$$

Now, imagine you are asked to factor the RHS of Equation 5.4. You might simply reverse the sequence of Equations 5.3 to 5.4. You would write

$$\begin{aligned} acx^2 + bcx + adx + bd &= cx(ax+b) + d(ax+b) \\ &= (ax+b)(cx+d). \end{aligned}$$

This is known as *factoring by grouping*.

Example 5.21

Factor $2b + 2c + ab + ac$ by using factoring by grouping.

Solution

$$\begin{aligned} 2b + 2c + ab + ac &= 2(b+c) + a(b+c) \\ &= (b+c)(2+a) \end{aligned}$$

which we rewrite in nicer style as

$$= (a+2)(b+c).$$

Example 5.22

Factor $a^2 + ab + 5ax + 5bx$ by using factoring by grouping.

Solution

$$\begin{aligned} a^2 + ab + 5ax + 5bx &= a(a+b) + 5x(a+b) \\ &= (a+b)(5x+a). \end{aligned}$$

Example 5.23

Factor $2yx - 5x + 6y - 15$ by using factoring by grouping.

Solution

$$\begin{aligned} 2xy - 5x + 6y - 15 &= x(2y-5) + 3(2y-5) \\ &= (x+3)(2y-5) \end{aligned}$$

Example 5.24

Factor $x^3y^2 + x^3 - 3y^2 - 3$ by using factoring by grouping.

Solution

$$\begin{aligned} x^3y^2 + x^3 - 3y^2 - 3 &= x^3(y^2+1) - 3(y^2+1) \\ &= (x^3-3)(2y+1) \end{aligned}$$

Example 5.25

Factor $6xy - 10x + 3y - 5$ by using factoring by grouping.

Solution

$$\begin{aligned} 6xy - 10x + 3y - 5 &= 2x(3y-5) + (3y-5) \\ &= (2x+1)(3y-5). \end{aligned}$$

In Examples 5.21 to 5.25 on pages 111–112 the terms were already in an order that made the factoring easy to spot. In the next sequence of examples the terms are not as conveniently ordered.

Example 5.26

Factor $x^2 - 32 - 8x + 4x$ by using factoring by grouping.

Solution

There is no factor common to the first two terms. There are rearrangements that will place a term with x next to the term x^2. Two such arrangements:

$$x^2 - 8x + 4x - 32$$

and

$$x^2 + 4x - 8x - 32.$$

These each lead to the same factorization. The first path is

$$\begin{aligned} x^2 - 8x + 4x - 32 &= x(x-8) + 4(x-8) \\ &= (x+4)(x-8). \end{aligned}$$

The second path is

$$\begin{aligned} x^2 + 4x - 8x - 32 &= x(x+4) - 8(x+4) \\ &= (x+4)(x-8). \end{aligned}$$

Example 5.27

Factor $3x^3 + 2y^3 + 2x^2y + 3xy^2$.

Solution

$$\begin{aligned} 3x^3 + 2y^3 + 2x^2y + 3xy^2 &= 3x^3 + 3xy^2 + 2y^3 + 2x^2y \\ &= 3x(x^2 + y^2) + 2y(y^2 + x^2) \\ &= (3x + 2y)(x^2 + y^2). \end{aligned}$$

Example 5.28

Factor $2xy^2 - 8y^2 + x - 4$.

Solution

$$\begin{aligned} 2xy^2 - 8y^2 + x - 4 &= 2y^2(x-4) + x - 4 \\ &= (x-4)(2y^2 + 1). \end{aligned}$$

Factorizations may typically be checked with little effort.Just expand the factored form. Let us check the answer of Example 5.28.

$$\begin{aligned} (x-4)(2y^2+1) &= 2y^2(x-4) + 1(x-4) \\ &= 2xy^2 - 8y^2 + x - 4. \checkmark \end{aligned}$$

Exercise 5.4

Factor by grouping.

1. $3ab+15a-4b-20$
2. $16xy+4xp+4py+p^2$
3. $3xy+4x-15y-20$
4. $5uv+25u+v+5$
5. $8xy+12x^2-6y-9x$
6. $10xy+25x^2-6y-15x$
7. $20uv+25u-12v^2-15v$
8. $4uv-u-12v+3$
9. $6ab-8a^2+9nb-12na$
10. $12xy-4x+9ny-3n$
11. $5bu-4bv-10xu+8xv$
12. $15mu-5mv-6nu+2nv$
13. $20xy-5x+16py-4p$
14. $3aw-9ak-2b^2w+6b^2k$
15. $2ab+a^2+2rb+ra$
16. $16xy-20x+12y-15$
17. $15xy-6x+20ky-8k$
18. $15uv-9u^4-25xv+15xu^3$
19. $3xy-6x^3+2vy-4vx^2$
20. $20az+15ac+8yz+6yc$
21. $24xy+32xn+9ny+12n^2$
22. $6mn+7m-30n-35$
23. $10xy+6x-5y-3$
24. $7xy+3x-56y^3-24y^2$
25. $4mn-10m^2-14n+35m$
26. $7xy+48y-42x-8y^2$
27. $21uv-25xu^2+35u^3-15xv$
28. $5xy+18+15x+6y$
29. $8uv-3+4u-6v$
30. $5xy-32+40x-4y$
31. $2xy-3x+2y-3$
32. $25xy-45x^4+15by-27bx^3$
33. $10xy-6x+5y-3$
34. $72ah-16ak^2+45bh-10bk^2$
35. $27xy+18x-21ay-14a$
36. $5p^2w^2+50p^2k+qw^2+10qk$
37. $7ab-a^2-14xb+2xa$
38. $35mn+15m+42n+18$
39. $6uv+u-6xv-x$
40. $56xy-40xp+21py-15p^2$
41. $70uv+7ur-20rv-2r^2$
42. $5xw-15xk-yw+3yk$
43. $14xy-49x-16y+56$
44. $50az+45ac-10x^2z-9x^2c$
45. $9xy-90x+7ny-70n$
46. $3xy+8x+21ny+56n$
47. $18xy-3x+6py-p$
48. $az+4ac+8xz+32xc$
49. $20uv-45u+32xv-72x$
50. $20xy-14x+10y-7$
51. $14xy+35x+2y+5$
52. $24uv+3u+40v+5$
53. $7xy+3x^3+7ny+3nx^2$
54. $90m^2h+81m^2k+50nh+45nk$
55. $30bz-54bc+25xz-45xc$
56. $63ab+90a+28b^2+40b$

5.7. Annoying cases made less annoying

Several pages and many exercise problems have passed since the mention of “annoying cases” on page 111. Recall the problem was to factor expressions in which many possibilities might have to be considered. The example mentioned was $12x^2 - 11x - 36$. It was discouraging that 12 has 6 divisors and 36 has 9 divisors.

Our goal in this section is to factor $Ax^2 + Bx + C$ even when A and C each have many factors.

Using factoring by grouping, we can work our way back from the sum

$$acx^2 + (bc + ad)x + bd$$

to the product

$$(ax + b)(cx + d).$$

But, how to get from

$$Ax^2 + Bx + C$$

to the form

$$acx^2 + (bc + ad)x + bd\,?$$

The answer to that question is the subject of this section.

When we expand $(ax + b)(cx + d)$ we obtain $acx^2 + (bc + ad)x + bd$. Typically we do not see the individual factors a, b, c and d, because they are combined into various numbers. The form we *see* is $Ax^2 + Bx + C$. For example, when we expand

$$(2x + 3)(5x + 7)$$

we see

$$10x^2 + 29x + 21,$$

where

$$10 = 2 \cdot 5, \quad 29 = 3 \cdot 5 + 2 \cdot 7, \quad \text{and} \quad 21 = 3 \cdot 7.$$

Figure 5.1 makes the correspondence $A = ac$, $B = bc + ad$, and $C = bd$ clear.

$$\begin{array}{c} Ax^2 \;\; + \;\; Bx \;\; + \;\; C \\ acx^2 + (bc + ad)x + bd \end{array}$$

FIGURE 5.1. Factors concealed and factors revealed.

We now make a crucial observation:

$$AC = bc \cdot ad \quad \text{and} \quad B = bc + ad.$$

Let us call bc "m" and call ad "n". That is, let

$$m = bc \quad \text{and} \quad n = ad.$$

Using these the names m and n,

$$AC = m \cdot n \tag{5.5}$$

and

$$B = m + n. \tag{5.6}$$

Since there are two equations 5.5 and 5.6 and two unknowns, we are guaranteed to find the values of m and n.

We cannot yet solve the pair of equations 5.5 and 5.6 algebraically. That must wait for a later time. But, think about the meaning of equations 5.5 and 5.6. They say that m and n are simply a pair of divisors of the product AC whose sum is B. That we can figure out.

Once we have determined the values of m and n, we replace B in $Ax^2 + Bx + C$ with $m + n$. That equation,

$$Ax^2 + (m+n)x + C$$

is one we factor using factoring by grouping.

Let us pause to summarize the strategy we have just planned.

Summary

To factor an expression given in the form $Ax^2 + Bx + C$,

(1) Find two divisors of $A \cdot C$ whose sum is B. Call them m and n.
(2) Write the expression $Ax^2 + (m+n)x + C$. Factor this expression using factoring by grouping.

Example 5.29

Factor $8x^2 + 34x - 9$.

Solution

$AC = 8(-9) = -72$, $B = 34$. We seek a pair of divisors of -72 whose sum is 34.

$$-72 = \begin{cases} -1 \cdot 72 & \\ -2 \cdot 36 & \text{Bingo!} \quad 36 - 2 = 34 \\ -3 \cdot 24 & \\ -4 \cdot 18 & \\ -6 \cdot 12 & \\ -8 \cdot 9 & \end{cases}$$

So, $m = 36, n = -2$. Then,

$$\begin{aligned} 8x^2 + \mathbf{36}x - \mathbf{2}x - 9 &= 4x(2x+9) - (2x+9) \\ &= (4x-1)(2x+9). \end{aligned}$$

Therefore, $8x^2 + 36x - 2x - 9 = (4x-1)(2x+9)$.

Remark 5.1

Yes. "Divisor" as used here appears to conflict with Definition 1.2 on page 3 which requires a divisor to be positive. But thinking of -2 as the additive inverse of the divisor 2 makes everything come out alright. Because the issue is so easily skirted, mathematicians would typically call -2 a divisor of -72 in this context.

Example 5.30

Factor $8x^2 + 22x + 9$.

Solution

$AC = 72, B = 22$. Find a pair of divisors of 72 whose sum is 22.

$$72 = \begin{cases} 1 \cdot 72 \\ 2 \cdot 36 \\ 3 \cdot 24 \\ 4 \cdot 18 & \text{Choose me!} \quad 18 + 4 = 22 \\ 6 \cdot 12 \\ 8 \cdot 9 \end{cases}$$

$$\begin{aligned} 8x^2 + 18x + 4x + 9 &= 8x^2 + 4x + 18x + 9 \\ &= 4x(2x+1) + 9(2x+1) \\ &= (4x+9)(2x+1). \end{aligned}$$

Although checks will no longer be made in the text, factorizations should be checked as in previous examples.

Therefore, $8x^2 + 18x + 4x + 9 = (4x+9)(2x+1)$.

Example 5.31

Factor $12x^2 - 35x + 18$.

Solution

$AC = 216, B = -35$. Find a pair of divisors of 216 whose sum is -35.

$$216 = \begin{cases} -1(-216) \\ -2(-108) \\ -3(-72) \\ -4(-54) \\ -6(-36) \\ -8(-27) \qquad \text{List no more divisor pairs. The answer is here!} \end{cases}$$

Then,

$$\begin{aligned} 12x^2 - 8x - 27x + 18 &= 12x^2 - 27x - 8x + 18 \\ &= 3x(4x-9) - 2(4x-9) \\ &= (3x-2)(4x-9). \end{aligned}$$

Therefore, $12x^2 - 35x + 18 = (3x-2)(4x-9)$. ■

The author admits that in the previous examples he has listed divisor pairs that did not have a chance of success. In the example just worked, why bother with the pair $(1, 216)$ when the difference of the divisors is only 35? The author will behave himself in subsequent examples.

Example 5.32

Factor $12x^2 - 11x - 36$. This the annoying expression from page 111.

Solution

$AC = -432$, $B = -11$. Find a pair of divisors of 432 whose sum is -11.

$$-432 = \begin{cases} \quad \vdots \\ 12(-36) \\ 16(-27) \quad \text{List no more. The answer is here!} \quad 16 - 27 = -11. \end{cases}$$

Then,

$$\begin{aligned} 12x^2 - 27x + 16x - 36 &= 3x(4x-9) + 4(4x-9) \\ &= (3x+4)(4x-9) \end{aligned}$$

Therefore, $12x^2 - 27x + 16x - 36 = (3x+4)(4x-9)$. ■

For Exercise 5.5, it might helpful to remember that every divisor of a number can be built from the number's divisors (Section 1.6 on page 17.) For example, every divisor of 12 is some combination of divisors $\{2, 2, 3\}$.

Suppose we want to know two divisors of 36 whose sum is 15. The prime factors of 36 are $\{2, 2, 3, 3\}$. With perhaps a little experimentation, we see that both factors of 2 and one of the 3s make 12.

$$\underbrace{2 \cdot 2 \cdot 3}_{12} \cdot 3.$$

The sum of 12 and the lone factor of 3 equals 15. So the divisors we seek are 12 and 3.

Example 5.33

Factor $6x^2 + 25x + 14$.

Solution

$AC = 2 \cdot 3 \cdot 2 \cdot 7$. We seek two collections of divisors from $\{2, 3, 2, 7\}$ whose sum is 25.

$$\underbrace{3 \cdot 7}_{21} \cdot \underbrace{2 \cdot 2}_{4}.$$

Then

$$\begin{aligned} 6x^2 + 25x + 14 &= 6x^2 + 21x + 4x + 14 \\ &= 3x(2x+7) + 2(2x+7) \\ &= (3x+2)(2x+7). \end{aligned}$$

Example 5.34

Factor $12x^2 - 11x - 36$. (Yes, this is the problem from page 111 again.)

Solution

$AC = 2 \cdot 2 \cdot 3 \cdot 2 \cdot 2 \cdot 3 \cdot 3$. We seek a pair of divisors from $\{2, 2, 3, 2, 2, 3, 3\}$ whose difference is 11.

$$\underbrace{2 \cdot 2 \cdot 2 \cdot 2}_{16} \cdot \underbrace{3 \cdot 3 \cdot 3}_{27}.$$

Then, as in Example 5.32 on the preceding page,

$$12x^2 - 27x + 16x - 36 = (3x+4)(4x-9).$$

Exercise 5.5

Factor.

1. $6n^2-5n-4$
2. $4n^2-9n-28$
3. $6x^2+13x+6$
4. $6n^2+31n+35$
5. $4a^2-27a+18$
6. $6r^2-49r+49$
7. $6k^2+7k+2$
8. $6x^2-7x-10$
9. $4n^2-12n+5$
10. $6r^2+17r+5$
11. $6m^2-m-35$
12. $4x^2+25x+6$
13. $4n^2+27n+18$
14. $6p^2+29p+28$
15. $4n^2+16n+15$
16. $6m^2-29m-42$
17. $6n^2-25n-25$
18. $6b^2+17b+12$
19. $6b^2-17b-14$
20. $6v^2-7v-20$
21. $6v^2-v-2$
22. $6x^2+13x-28$
23. $6n^2-19n+3$
24. $4v^2+15v+9$
25. $6b^2+7v+2$
26. $4a^2-23a+28$
27. $6n^2+17n-14$
28. $6x^2-25x+4$
29. $6a^2+17a+7$
30. $6v^2-v-15$
31. $10r^2-71r+7$
32. $10n^2+53n+36$
33. $12x^2+7x-10$
34. $9x^2-28x+3$
35. $9b^2-101b+110$
36. $9b^2-49b+20$
37. $9x^2+106x+77$
38. $12x^2+53x+56$
39. $12x^2+40x-63$
40. $8p^2+10p-33$
41. $9n^2-104n+55$
42. $9m^2-70m+49$
43. $12m^2-7m-10$
44. $10n^2+123n+36$
45. $9m^2+56m+12$
46. $12x^2-25x-50$
47. $10p^2+99p-10$
48. $8r^2-37r-15$
49. $9x^2+40x+16$
50. $6n^2-31n+18$
51. $12n^2-13n-90$
52. $9r^2-36r+32$
53. $10x^2-11x-35$
54. $9n^2-62n-80$
55. $9n^2+91n+90$
56. $12x^2-67x+90$
57. $9m^2-29m+22$
58. $9p^2-55p+50$
59. $9x^2-85x+36$
60. $10x^2-x-24$

5.8. Greatest common factor

The heading of this section, “Factor out greatest common factor”, is simple advice that will sometimes make your effort far less than it might be otherwise.

Suppose you wish to factor $12x^2+60x+72$. Eek! So many possibilities. But, look before you leap. Notice that 12 is a common factor of all three terms. Before you do anything else, factor out the common factor of 12. Now look how easy the task has become.

$$12x^2+60x+72 = 12[x^2+5x+6].$$

Feel better now? You should, because factoring x^2+5x+6 is nearly automatic.

$$12x^2+60x+72 = 12(x+2)(x+3).$$

Moral of story: always, *always*, factor out any common factor before you do anything else.

Example 5.35

Factor $36x^2-108x+72$.

Solution

Maybe it is not obvious that 36 is a factor of 108. It is worth the trouble to find out, because if you can factor out 36, the work will be ever so lighter. In fact, $36\cdot 3 = 108$. So,

$$\begin{aligned} 36x^2-108x+72 &= 36[x^2-3x+2] \\ &= 12(x-1)(x-2). \end{aligned}$$

Did we mention to always factor out any common factor first? Oh, so we did, yes.

Exercise 5.6

Factor. Hint: every expression has a factor common to all three terms.

1. $4r^2+20r-96$
2. $3n^2-12n-36$
3. $3n^2-12n-15$
4. $4x^2+36x-40$
5. $3x^2-12x-180$
6. $4v^2+32v+28$
7. $4x^2+28x+40$
8. $2b^2-6b+4$
9. $5x^2-5x-60$
10. $3n^2-45n+150$
11. $5n^2+55n+150$
12. $5r^2-90r+400$
13. $5x^2-65x+150$
14. $3r^2+18r-21$
15. $3n^2+27n+42$
16. $5n^2-65n+200$
17. $2x^2-32x+126$
18. $3x^2-39x+120$
19. $5p^2+10p-40$
20. $6m^2+66m+108$
21. $4p^2-16p-84$
22. $2m^2-32m+126$
23. $4r^2+32r+48$
24. $6m^2+18m-240$
25. $6x^2+72x+162$
26. $2x^2+28x+96$
27. $3r^2+15r-108$
28. $3n^2+45n+150$
29. $2x^2-14x-16$
30. $6a^2+36a-162$
31. $7x^3-30x^2+27x$
32. $14x^3+4x^2-18x$
33. $5x^3-26x^2+24x$
34. $6v^4+2v^3-4v^2$
35. $30x^3+204x^2+270x$
36. $21v^3-186v^2-240v$

5.9. Special forms

These forms are almost instantly factorable. They turn up with surprising frequency. Special forms are your friends. You want to be able to recognize them in a crowd. The three special forms are

Square of sum: $a^2 + 2ab + b^2 = (a+b)^2$.
Square of difference: $a^2 - 2ab + b^2 = (a-b)^2$.
Difference of squares: $a^2 - b^2 = (a-b)(a+b)$.

5.9.1. Square of a sum

When we expand $(a+b)^2$ we obtain $(a+b)(a+b) = a^2 + 2ab + b^2$. This trinomial is called the *square of a sum*. It is a "perfect square trinomial". When you meet the square of a sum, you can instantly factor it using Equation 5.7 which is true for all numbers a and b.

(5.7) **Square of sum** $a^2 + 2ab + b^2 = (a+b)^2$.

Note that in Equation 5.7, the factor 2 of the middle term will appear regardless of the values of a and b. In other words, the factor of 2 is an inherent feature of the form. You will pick the square of a sum out of a crowd if you look for a trinomial whose first and last terms are perfect squares and whose middle term is twice the product of a and b.

5.9.2. Square of difference

When we expand $(a-b)^2$ we obtain $(a-b)(a-b) = a^2 - 2ab + b^2$. This trinomial is called the *square of a difference*. It is a "perfect square trinomial". Equation 5.8 which is true for all numbers a and b shows the factorization of the square of a difference.

(5.8) **Square of difference** $a^2 - 2ab + b^2 = (a-b)^2$.

In Equation 5.8, the factor 2 of the middle term will appear regardless of the values of a and b. The first and last terms are perfect squares and the middle term is twice the product of a and b.

The square of a sum and the square of a difference are identical except for the sign of the middle term.

5.9.3. Difference of squares

Expand $(a+b)(a-b)$ produces $a^2 - b^2$. This binomial is called the *difference of squares*. Equation 5.9 which is true for all numbers a and b shows its factorization.

(5.9) **Difference of squares** $a^2 - b^2 = (a+b)(a-b)$.

You will be surprised at how often the difference of squares appears. Recognizing it is often the key to successfully working a problem or proof.

5.9.4. Sum of squares

The "sum of squares" $x^2 + y^2$ does *not* factor. It is easy to see why.

Let a, b, c, d, x, y represent any numbers. Suppose $x^2 + y^2$ factors. Then

$$\begin{aligned} x^2 + y^2 &= (ax + by)(cx + dy) \\ &= acx^2 + (ad + bc)xy + bdy^2. \end{aligned}$$

But, a and c are either both negative or both positive. And, b and d are either both negative or both positive. This means ad and bc are either both negative or both positive. In either case, the sum $ad + bc$ cannot equal 0.

5.9.5. Pictorial illustrations

No book that includes factorization is complete without the pictorial proof in Figure 5.2.

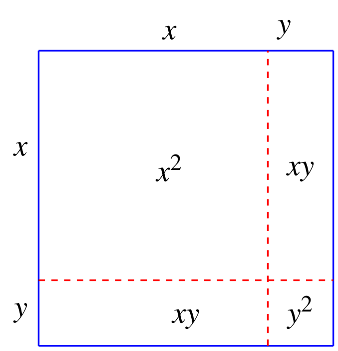


FIGURE 5.2. $(x + y)^2 = x^2 + 2xy + y^2$.

The area of the large square is equal to $(x + y)^2$ and equal to $x^2 + 2xy + y^2$. Similar illustrations can be given for $(x - y)^2$ and $(x + y)(x - y)$. Figure 5.2 and others like it were known as early as the middle of the 8th century.

5.9.6. Using special forms

Now that you have met your new friends, let us practice recognizing them.

Example 5.36

Factor $9a^2 + 6ab + b^2$.

Solution

Noticing that the first term is a perfect square, $(3a)^2$, and the last term is a perfect square, b^2, we begin to hope this is a familiar form. Once we write the middle term as $2(3a \cdot b)$, we know this is the square of a sum.

$$\begin{aligned} 9a^2+6ab+b^2 &= (3a)^2+2(3a)(b)+b^2 \\ &= (3a+b)^2. \\ \therefore\ 9a^2+6ab+b^2 &= (3a+b)^2. \end{aligned}$$

Example 5.37

Factor $a^2+4ab+4b^2$.

Solution

First term is a^2. Last term is $(2b)^2$. Middle term is $2(a \cdot 2b)$. This is a friendly form.

$$\begin{aligned} a^2+4ab+4b^2 &= a^2+2a(2b)+(2b)^2 \\ &= (a+2b)^2. \\ \therefore\ a^2+4ab+4b^2 &= (a+2b)^2. \end{aligned}$$

Example 5.38

Factor $25a^2+70ab+49b^2$.

Solution

First term is $(5a)^2$. Last term is $(7b)^2$. Middle term is $2(5a \cdot 7b)$. So, square of sum.

$$\begin{aligned} 25a^2+70ab+49b^2 &= (5a)^2+2(5a)(7b)+(7b)^2 \\ &= (5a+7b)^2. \\ \therefore\ a25a^2+70ab+49b^2 &= (5a+7b)^2. \end{aligned}$$

Example 5.39

Factor $4a^2-12ab+9b^2$.

Solution

First term is $(2a)^2$. Last term is $(3b)^2$. Middle term is $-2(2a \cdot 3b)$. So, square of difference.

$$\begin{aligned} 4a^2-12ab+9b^2 &= (2a)^2-2(2a)(3b)+(3b)^2 \\ &= (2a-3b)^2. \\ \therefore\ 4a^2-12ab+9b^2 &= (2a-3b)^2. \end{aligned}$$

Example 5.40

Factor $x^2 - 12x + 36$.

Solution

First term is x^2. Last term is 6^2. Middle term is $-2(x \cdot 6)$. So, square of difference.

$$\begin{aligned} x^2 - 12x + 36 &= x^2 - 2(x)(6) + 6^2 \\ &= (x-6)^2. \\ \therefore\ x^2 - 12x + 36 &= (x-6)^2. \end{aligned}$$

Example 5.41

Factor $x^2 - 4$.

Solution

First term is x^2. Last term is 2^2. No middle term. So, difference of squares.

$$\begin{aligned} x^2 - 4 &= x^2 - 2^2 \\ &= (x-2)(x+2). \\ \therefore\ x^2 - 4 &= (x-2)(x+2). \end{aligned}$$

Example 5.42

Factor $36x^2 - 25y^2$.

Solution

First term is $(6x)^2$. Last term is $(5y)^2$. No middle term. So, difference of squares.

$$\begin{aligned} 36x^2 - 25y^2 &= (6x)^2 - (5y)^2 \\ &= (6x-5y)(6x+5y). \\ \therefore\ 36x^2 - 25y^2 &= (6x-5y)(6x+5y). \end{aligned}$$

Example 5.43

Factor $20a^2 - 100a + 125$.

Solution

Remember to factor out any common factor before doing anything else!

$$20a^2 - 100a + 125 = 5[4a^2 - 20a + 25].$$

The expression in brackets is the square of a difference. Thus,

$$= 5(2a-5)^2$$

$$\therefore\ 20a^2 - 100a + 125 = 5(2a-5)^2.$$

Example 5.44

Factor $8u^2 - 72$.

Solution

This looks non-factorable. The only candidate is the difference of squares, but neither 8 nor 72 are square numbers. But, take out the common factor of 8 and the rest is trivial.

$$\begin{aligned} 8u^2 - 72 &= 8[x^2 - 9]. \\ &= 8(x-3)(x+3). \\ \therefore\ 8u^2 - 72 &= 8(x-3)(x+3). \end{aligned}$$

In Example 5.44, factoring out 8 does not merely make the factorization easier, it makes it possible.

Exercise 5.7

Factor.

1. $x^2+12x+36$
2. $25x^2-36$
3. $16b^2-56b+49$
4. $4m^2-4m+1$
5. $25a^2-60a+36$
6. $64n^2-208n+169$
7. $9x^2+78x+169$
8. $49b^2-4$
9. $196p^2-1$
10. $49x^2-169$
11. $49x^2-140x+100$
12. $25v^2+60v+36$
13. $25p^2-40p+16$
14. $25p^2-121$
15. $9k^2+48k+64$
16. $4x^2+12x+9$
17. $49n^2+140n+100$
18. $81b^2-100$
19. $100v^2-60v+9$
20. $9r^2-66r+121$
21. $16k^2-8k+1$
22. $16a^2-9$
23. $16n^2-24n+9$
24. x^2-25
25. $9a^2+12a+4$
26. $25n^2+110n+121$
27. $25r^2+90r+81$
28. $121x^2-1$
29. $64k^2-25$
30. $100v^2+60v+9$
31. $81v^2-180v+100$
32. $16n^2-40n+25$
33. $81m^2+72m+16$
34. $49r^2+28r+4$
35. $25n^2+80n+64$
36. $169x^2-196$
37. $4m^2+52m+169$
38. $36n^2-60n+25$
39. $100m^2+20m+1$
40. $64a^2+176a+121$
41. $144x^2+264x+121$
42. $81x^2-49$
43. $64r^2-169$
44. $49x^2+56x+16$
45. $169x^2+260x+100$
46. $49x^2-70x+25$
47. $36n^2-169$
48. $121x^2+176x+64$
49. $4x^2+28x+49$
50. $64x^2-80x+25$
51. $169x^2-156x+36$
52. $81a^2-252a+196$
53. $r^2-22r+121$
54. $81m^2+18m+1$
55. $121b^2-144$
56. $9b^2-6b+1$
57. $16x^2+40x+25$
58. $9n^2-84n+196$

5.10. Expressions quadratic in form

Once you understand some mathematics well, you will often be able to find your way around in an *unfamiliar* mathematical context. You will find it is a little like searching the faces in an unfamiliar crowd hoping to find someone you know. This section will give you the experience of doing that.

Example 5.45

Factor $(x+3)^2+5(x+3)+6$.

Solution

There is a friendly form lurking in the expression $(x+3)^2+5(x+3)+6$. Substituting u for $x+3$ makes the form of the expression clear. It is u^2+5u+6. Then,

$$u^2+5u+6=(u+3)(u+2)$$

Back substitute $x+3$ for u,

$$=(x+3+3)(x+3+2).$$

$\therefore\ (x+3)^2+5(x+3)+6=(x+6)(x+5)$.

Example 5.46

Factor $(2x+5)^2+6(2x+5)+9$.

Solution

Substitute u for $2x+5$,

$$\begin{aligned}(2x+5)^2+6(2x+5)+9&=u^2+6u+9\\&=(u+3)^2.\end{aligned}$$

Back substitute $2x+5$ for u,

$$=(2x+5+3)^2.$$

$\therefore\ (2x+5)^2+6(2x+5)+9=(2x+8)^2$.

Example 5.47

Factor x^4-25.

Solution

Note that $x^4 = (x^2)^2$. Then,

$$\begin{aligned} x^4 - 25 &= (x^2)^2 - 25 \\ &= (x^2+5)(x^2-5). \end{aligned}$$

$\therefore\ x^4 - 25 = (x^2+5)(x^2-5)$.

Example 5.48

Factor $x^4 - 81$.

Solution. Note that $x^4 = (x^2)^2$. Then,

$$\begin{aligned} x^4 - 81 &= (x^2)^2 - 81 \\ &= (x^2+9)(x^2-9) \end{aligned}$$

But $x^2 - 9$ is the difference of squares, so

$$= (x^2+9)(x+3)(x-3).$$

$\therefore\ x^4 - 81 = (x^2+9)(x+3)(x-3)$.

Examples 5.45 to 5.48 on pages 129–130 all involved substituting a simple name, like "u", for a complex name, like "$x+3$". in order to make a familiar form obvious.

The next example also depends on recognizing a familiar form. But, the familiar form only appears after we do a little work.

Example 5.49

Factor $x^2 + 10x + 25 - 4y^2$.

Solution

The key is to pick out the perfect square trinomial $x^2 + 10x + 25$. Then,

$$x^2 + 10x + 25 - 4y^2 = (x+5)^2 - 4y^2.$$

This is the familiar difference of squares. So,

$$= (x+5+2y)(x+5-2y).$$

$\therefore\ x^2 + 10x + 25 - 4y^2 = (x+2y+5)(x-2y+5)$.

Example 5.50

Factor $x^2 + 6x + 9 + 5(x+3) + 6$.

Solution

The first three terms, x^2+6x+9, factor as $(x+3)^2$. Then,

$$\begin{aligned}(x+3)^2+5(x+3)+6 &= [(x+3)+2][(x+3)+3]\\ &= (x+5)(x+6).\end{aligned}$$

$\therefore\ x^2+6x+9+5(x+3)+6=(x+5)(x+6)$.

Example 5.51

Factor $x^2-14x+49-81$.

Solution

Notice that $81=9^2$ and that $x^2-14x+49$ is the square of a difference $x-7$. Then,

$$x^2-14x+49-81=(x-7)^2-9^2.$$

The difference of squares. So,

$$=(x-7+9)(x-7-9).$$

$\therefore\ x^2-14x+49-81=(x+2)(x-16)$. ■

Example 5.51 probably seemed a little artificial, because the "$49-81$" part would be more likely to appear as "-32". The next example uses "-32" instead of "$49-81$" .

Example 5.52

Factor $x^2-14x-32$.

Solution

Rewrite $x^2-14x-32$ as $x^2-14x+49-81$. Now proceed as in the previous example to conclude that

$$x^2-14x-32=(x+2)(x-16).$$

■

If Example 5.51 was like picking a familiar face out of a crowd, Example 5.52 is like playing a game of Hide and Seek.

Just as experience with Example 5.51 provided insight into Example 5.52, so too will your experience with this section of the book provide insight in future mathematical contexts.

Example 5.53

Factor $x^2+4x-12$.

Solution

The first two terms x^2+4x remind us of the perfect square trinomial x^2+4x+4. Wishing to have this, we imagine rewriting -12 as $4-16$. When we see the -16 appear, we recognize the difference of squares.

$$\begin{aligned} x^2+4x-12 &= x^2+4x+4-16 \\ &= (x+2)^2-16 \\ &= [(x+2)+4][(x+2)-4]. \end{aligned}$$

$\therefore\ x^2+4x-12=(x+6)(x-2).$ ■

You may be wondering "Who would ever think to rewrite the -12 as $4-16$?" The answer is: you! With experience, you will quickly you "see" strategies like this.

In Example 5.54 success depends on introducing a new term.

Example 5.54

Factor $x^4+x^2y^2+y^4$.

Solution

Notice that

$$x^4+x^2y^2+y^4$$

is *almost* a perfect square. If only it were

$$x^4+2x^2y^2+y^4,$$

because

$$x^4+2x^2y^2+y^4=(x^2+y^2)^2.$$

We may add x^2y^2 providing we subtract x^2y^2. That amounts to adding 0.

$$\begin{aligned} x^4+x^2y^2+y^4 &= x^4+x^2y^2+y^4+x^2y^2-x^2y^2 \\ &= x^4+2x^2y^2+y^4-x^2y^2 \\ &= (x^2+y^2)^2-x^2y^2 \\ &= (x^2+y^2)^2-(xy)^2 \\ &= [(x^2+y^2)+xy][(x^2+y^2)-xy]. \end{aligned}$$

$\therefore\ x^4+x^2y^2+y^4=(x^2+xy+y^2)(x^2-xy+y^2).$

The last example may seem like playing Hide and Seek against a stealth opponent.

Exercise 5.8

Factor.

For questions 1 - 6, see Examples 5.45 - 5.48.

1. $(x+7)^2-3(x+7)-10$
2. $(x-1)^2+(x-1)-2$
3. y^4-9
4. x^4-16
5. $81a^4-16$
6. $16x^4-1$

For questions 7 - 12, see Examples 5.49 - 5.50.

7. $(x^2-2x+1)-25$
8. $x^2-14x+49-9y^2$
9. $4x^2+36x+81-16y^2$
10. $a^2-16a+64-100b^2$
11. $36a^2-84a+49-121b^2$
12. $x^2-y^2-4yz-4z^2$

For questions 13 - 14, see Examples 5.51 - 5.52.

13. $9x^2+30x+16$
14. $64x^4+1$

The following use ideas from several examples.

15. a^2-b^2+a-b
16. $(2x-3)^2-(x+2)^2$
17. x^2-y^2-4y-4
18. $x^2(x-9)+9(9-x)$
19. $(a-b)^2+a-b$

5.11. Polynomials with rational coefficients

Polynomials with rational coefficients can be handled as in Example 5.55.

Example 5.55

Factor $\frac{1}{6}x^2 - \frac{1}{6}x - 1$.

Solution

Multiply by $1 = \frac{6}{6}$.

$$\begin{aligned}\frac{1}{6}x^2 - \frac{1}{6}x - 1 &= \frac{6}{6}\left(\frac{1}{6}x^2 - \frac{1}{6}x - 1\right)\\ &= \frac{1}{6}(x^2 - x - 6)\\ &= \frac{1}{6}(x-3)(x+2).\end{aligned}$$

5.12. Polynomials with irrational coefficients

A polynomial with irrational coefficients may be rewritten as a product. No new techniques are needed.

Example 5.56

Rewrite as a product. $x^2 - 2\sqrt{3} + 3$.

Solution

Since $3 = (\sqrt{3})^2$, we can write 3 as the square of the number $\sqrt{3}$. Then,

$$x^2 - 2\sqrt{3} + (\sqrt{3})^2 = (x - \sqrt{3})^2.$$

Example 5.57

Rewrite as a product. $x^2 - \sqrt{3}x - \sqrt{2}x + \sqrt{6}$.

Solution

The technique of factoring by grouping comes to our rescue.

$$\begin{aligned}x^2 - \sqrt{3}x - \sqrt{2}x + \sqrt{6} &= x(x - \sqrt{3}) - \sqrt{2}(x - \sqrt{3})\\ &= (x - \sqrt{2})(x - \sqrt{3}).\end{aligned}$$

Exercise 5.9

Rewrite each sum as a product.

1. $\frac{1}{6}x^2 - \frac{2}{3}x + \frac{1}{2}$
2. $x^2 + \frac{5}{3}x + \frac{2}{3}$
3. $x^2 + \frac{1}{4}x - \frac{1}{8}$
4. $x^2 + x + \frac{1}{4}$
5. $x^2 - \frac{1}{25}$
6. $\frac{x^2}{3} + \frac{x}{3} + \frac{1}{12}$
7. $x^2 + \sqrt{3}x - \sqrt{2}x - \sqrt{6}$
8. $x^2 - \sqrt{5}x - \sqrt{2}x + \sqrt{10}$
9. $x^2 - \sqrt{2}x - x + \sqrt{2}$
10. $x^2 - 2\sqrt{2}x + 2$
11. $x^2 + 2\sqrt{5}x + 5$
12. $x^2 - 5$
13. $x^2 - 7$
14. $x^2 - \sqrt{11}x + \sqrt{7}x - \sqrt{77}$
15. $x^2 - \sqrt{7}x - \sqrt{2}x + \sqrt{14}$
16. $x^2 - 2\sqrt{2}x - 2x + 4\sqrt{2}$

Chapter 6

Rational expressions

A rational expression is a fraction in which the numerator and denominator are polynomials. That is, expressions of the form

$$\frac{\text{polynomial}}{\text{polynomial}}.$$

You have a lot of experience with the special case in which the degree of both the numerator and denominator is 0. The fraction $\frac{3}{5}$ is a zero degree polynomial over a zero degree polynomial.

This chapter is short and contains no new ideas. It consists of examples and exercises for you to gain experience working with rational expressions.

Example 6.1 (Simplification)

Simplify $\frac{x^2+5x+6}{x^2-2x-15}$.

Solution

Factors common to both numerator and denominator cancel. Just like always. And, just like always, seeing the numerator and denominator as products of factors is the key.

$$\frac{x^2+5x+6}{x^2-3x-10} = \frac{(x+2)(x+3)}{(x+2)(x-5)} = \frac{(x+3)}{(x-5)}.$$

■

The cancellation is valid only if there is no division by 0. So, we should have said

$$\frac{x^2+5x+6}{x^2-3x-10} = \frac{(x+2)(x+3)}{(x+2)(x-5)} = \frac{(x+3)}{(x-5)}. \quad \text{Providing } x \neq -2.$$

We are not going to accompany each such calculation with a "providing . . . " clause. Instead, it will be assumed that the domain is suitably restricted so that expressions are meaningful. For example, when we write $\frac{(x-4)(x+3)}{(x-7)(x+1)}$ it is assumed that $x \neq 7, x \neq -1$.

You know that when multiplying fractions, canceling before computing the product avoids unnecessary computation. This is still true.

Example 6.2 (Multiplication)

Simplify

$$\frac{x^2+2x-3}{x^2+3x-10} \cdot \frac{x^2+9x+14}{x^2-x-12}.$$

Solution.

$$\begin{aligned}
\frac{x^2+2x-3}{x^2+3x-10} \cdot \frac{x^2+9x+14}{x^2-x-12} &= \frac{(x-1)(x+3)}{(x+5)(x-2)} \cdot \frac{(x+2)(x+7)}{(x+3)(x-4)} \\
&= \frac{x-1}{(x+5)(x-2)} \cdot \frac{(x+2)(x+7)}{x-4} \\
&= \frac{(x-1)(x+2)(x+7)}{(x+5)(x-2)(x-4)}.
\end{aligned}$$

We usually leave the answer in factored form.

Example 6.3 (Division)

$$\frac{x^2-6x+8}{x^2-x-2} \div \frac{5-x}{x^2-4x-5}.$$

Solution.

$$\begin{aligned}
\frac{x^2-6x+8}{x^2-x-2} \div \frac{5-x}{x^2-4x-5} &= \frac{x^2-6x+8}{x^2-x-2} \cdot \frac{x^2-4x-5}{5-x} \\
&= \frac{(x-4)(x-2)}{(x-2)(x+1)} \cdot \frac{(x-5)(x+1)}{5-x} \\
&= \frac{(x-4)(x-2)}{(x-2)(x+1)} \cdot \frac{-(x-5)(x+1)}{x-5} \qquad (6.1) \\
&= (x-4)(-1) \\
&= -(x-4) \\
&= 4-x
\end{aligned}$$

■

Be sure you understand how Equation 6.1 was obtained. This was discussed in *Beginning Algebra*. By way of quick review,

$$\begin{aligned}\frac{a}{c-b} &= \frac{-1}{-1}\cdot\frac{a}{c-b},\ c\neq 0\\ &= \frac{-a}{-1(c-b)}\\ &= \frac{-a}{b-c}.\end{aligned}$$

Exercise 6.1

Simplify

1. $\dfrac{3x+12}{x^2-16}$
2. $\dfrac{5x^2-40x}{x^2-64}$
3. $\dfrac{5v^2-30v}{v^2-5v-6}$
4. $\dfrac{b^2-18b+81}{b^2-4b-45}$
5. $\dfrac{k^2+12k+27}{k^2-4k-21}$
6. $\dfrac{-p^2+10p-21}{p^2-6p+9}$
7. $\dfrac{k^2-3k+2}{k^2+4k-12}$
8. $\dfrac{-x^2+9x-8}{x^2+4x-5}$
9. $\dfrac{p^2-2p-24}{p^2+14p+40}$
10. $\dfrac{x^2-5x+6}{x^2+7x-18}$

Multiply or divide as indicated

11. $\dfrac{v^2-v-56}{6v+42}\cdot\dfrac{6v+48}{64-8v}$
12. $\dfrac{4b-8}{4b+32}\cdot\dfrac{b^2+15b+56}{2b^2-4b}$
13. $\dfrac{r^2-12r+32}{r+5}\cdot\dfrac{r^2+3r-10}{3r-6}$
14. $\dfrac{x^2-6x+8}{x^2-x-2}\div\dfrac{5-x}{x^2-4x-5}$
15. $\dfrac{x^2-8x+16}{x^2-6x+9}\cdot\dfrac{8x-24}{x^2-8x+16}$
16. $\dfrac{4x-12}{20x-16}\cdot\dfrac{40x-32}{8x+64}$
17. $\dfrac{56m^2}{3m+21}\cdot\dfrac{m^2+3m-28}{m-4}$
18. $\dfrac{m^2-49}{2m^3+14m^2}\cdot\dfrac{2m^3-8m^2}{m-4}$
19. $\dfrac{5n^3-20n^2}{4}\div\dfrac{n^2-12n+32}{4n-20}$
20. $\dfrac{7k+49}{5k-15}\div\dfrac{k^2+4k-21}{5k-15}$

Example 6.4 (Addition and subtraction)

Simplify $\frac{y}{x^2+xy}+\frac{2x}{x^2-y^2}$.

Solution

Same as always. A common denominator is required. And, same as always, seeing the denominators as products of factors is the key to discovering the most convenient common denominator.

$$\begin{aligned}\frac{y}{x^2+xy}+\frac{2x}{x^2-y^2} &= \frac{y}{x(x+y)}+\frac{2x}{(x+y)(x-y)}\\ &= \frac{y(x-y)}{x(x+y)(x-y)}+\frac{2x^2}{x(x+y)(x-y)}\\ &= \frac{xy-y^2+2x^2}{x(x+y)(x-y)}.\end{aligned}$$

The factorization of $xy-y^2+2x^2$ is easier to see if we write (or imagine) $2x^2+xy-y^2$.

This simplifies,

$$\begin{aligned}&= \frac{(2x-y)(x+y)}{x(x+y)(x-y)}\\ &= \frac{2x-y}{x(x-y)}.\end{aligned}$$

Example 6.5 (Addition and subtraction)

Simplify $\frac{x^2}{x^2+2xy+y^2}+\frac{x}{y+x}$.

Solution.

$$\begin{aligned}\frac{x^2}{x^2+2xy+y^2}+\frac{x}{y+x} &= \frac{x^2}{(x+y)^2}+\frac{x}{y+x}\\ &= \frac{x^2}{(x+y)^2}+\frac{x(x+y)}{(x+y)^2}\\ &= \frac{x^2+x^2+xy}{(x+y)^2}\\ &= \frac{2x^2+xy}{(x+y)^2}\end{aligned}$$

Exercise 6.2

Simplify. Leave denominator factored. Expand numerator.

1. $\frac{a+1}{5a-2}-\frac{2}{5}$
2. $\frac{a-5}{a+4}+\frac{2}{3a+6}$
3. $\frac{5n}{2}+\frac{6}{3n-15}$
4. $\frac{4b}{2b}-\frac{5b-6}{2b-12}$
5. $\frac{6x-4}{2}-\frac{x-1}{x+1}$
6. $\frac{2}{k-5}+\frac{5k}{2k+4}$
7. $\frac{6}{3k}-\frac{4k+4}{15k^2-9k}$
8. $\frac{3}{n-4}+\frac{4}{n-3}$
9. $\frac{2r}{r-s}+\frac{3r}{r+3}$
10. $\frac{2}{b-4}+\frac{3}{b-1}$
11. $\frac{4}{x-2}+\frac{3x}{x-3}$
12. $\frac{2m-5}{2m^2-6m}-\frac{6}{3m}$
13. $\frac{6}{5}-\frac{2p+6}{p-3}$
14. $\frac{3}{v+1}-\frac{2v}{v-2}$
15. $\frac{5}{3n-5}-\frac{4n}{6}$
16. $\frac{4}{x+4}-\frac{6x-5}{3x-1}$
17. $\frac{5}{3}-\frac{5x-3}{x-3}$
18. $\frac{2}{2v^2-12v}-\frac{4}{2v}$
19. $\frac{5}{2}+\frac{n-5}{n-4}$
20. $\frac{3x}{x-3}-\frac{4x}{x-2}$

Exercise 6.3

Simplify.

1. $\frac{1}{p-1}+\frac{1}{1-p}$
2. $1-\frac{p}{1+p}$
3. $\frac{6}{3n^2-12n}+\frac{2}{2n}$
4. $\frac{4a}{a+5}-\frac{4a}{5a-2}$
5. $\frac{5b}{3b}+\frac{6}{b+1}$
6. $\frac{6}{2n}-\frac{5}{6n^2-4n}$
7. $\frac{6}{a+1}-\frac{3}{6a+5}$
8. $\frac{6}{x-5}-\frac{4x}{x+5}$
9. $\frac{2x}{3x}+\frac{x-3}{2x^2+8x}$
10. $3r-\frac{5r}{r^2+7r+12}$
11. $\frac{2}{3k+2}-\frac{3}{3k+3}$
12. $\frac{x+4}{x-5}-\frac{2x}{x-1}$
13. $\frac{3x-1}{x-4}-\frac{5}{2x+6}$
14. $\frac{k-1}{3k^2+24k+45}+\frac{6}{3k}$
15. $\frac{2n}{5}+\frac{5}{n^2-2n-24}$
16. $\frac{2}{r-1}-\frac{5}{r+2}$
17. $\frac{5}{2x}-\frac{x-5}{x^2+8x+16}$
18. $\frac{x-4}{x^2+3x-18}-\frac{3x}{6}$
19. $\frac{3}{3n-2}+\frac{6}{n-2}$
20. $\frac{6x}{6x^3-54x^2+108x}+3$
21. $\frac{5}{x^2+12x+12}+\frac{6x}{2x}$
22. $2p\frac{p+4}{p^2+4p-12}$
23. $\frac{3n}{3n+12}+\frac{5}{n-1}$
24. $\frac{2r}{r-2}+\frac{6}{6r^2+18r}$
25. $\frac{4}{20b^2+20b}+\frac{3}{b-6}$
26. $\frac{b+1}{2b^2-10b-12}+\frac{6b}{3}$
27. $\frac{x+2}{x^2-5x+4}-\frac{2}{5x^2}$
28. $\frac{r+1}{3r^2+6r-72}-\frac{5r}{3r}$
29. $\frac{4}{5}-\frac{n-5}{n^2-9}$
30. $\frac{6v}{2}-\frac{v-4}{v^2-5v+4}$
31. $\frac{2}{2p}+\frac{6}{2p^2-14p+24}$
32. $\frac{3}{6x+30}+\frac{5}{x+5}$

Exercise 6.4

Simplify

1. $\dfrac{x-5}{x-1}+\dfrac{x-9}{x^2-4x+3}$
2. $\dfrac{x-3}{x+3}-\dfrac{5}{x-3}$
3. $\dfrac{x-7}{x+7}-\dfrac{x-7}{3x+21}$
4. $\dfrac{x^2-x-6}{x^2-49}\cdot\dfrac{x^2-14x+49}{x^2-2x-3}$
5. $\dfrac{x^2+6x-7}{x^2+4x+4}\cdot\dfrac{x^2+x-2}{x^2-2x+1}$
6. $\dfrac{x^2-1}{x^4-81}\cdot\dfrac{x^2-9}{x+1}$
7. $\dfrac{x^2-2x-15}{-x^2+4x+5}\cdot\dfrac{-x^2+7x-10}{-x^2+2x+15}$
8. $\dfrac{2x^2+3x+1}{x^2-x-2}\cdot\dfrac{-x^2+7x-10}{2x^2+5x+2}$
9. $\dfrac{4x^2+4x+1}{x^2+2x-35}\cdot\dfrac{x^2-6x+5}{2x^2-x-1}$
10. $\dfrac{x^2+6x+9}{x^2+2x-35}\cdot\dfrac{x^2+5x-14}{x^2-9}$
11. $\dfrac{x^4-1}{x^2+6x-7}\cdot\dfrac{x^2+5x-14}{x^2-4x-5}$
12. $\dfrac{x^2+8x+16}{x^2+4x-21}\cdot\dfrac{x^2-8x+15}{x^2-x-20}$
13. $\dfrac{x^2-10x+25}{x^2-7x+10}\cdot\dfrac{x^2+4x-12}{x^2+x-30}$
14. $\dfrac{x^2-36}{x^2-4x+4}\cdot\dfrac{x^2-4}{x^2-10x+24}$
15. $\dfrac{x^2+8x-9}{x^2-81}\cdot\dfrac{x^2-10x+9}{x^2+8x-9}$
16. $\dfrac{(x+2)^2-(x+2)-6}{x^2-2x+1}\cdot\dfrac{x^2+4x-5}{x^2+8x+16}$

Chapter 7

Quadratic Equations

A polynomial equation has a polynomial on one side and a constant on the other. For example, the linear equations $2x+3=5$ and $5x-8=0$ are each polynomial equations. The equations $2x^2+7x+6=0$ and $x^3-x^2+5x+7=0$ are also polynomial equations. The degree of a polynomial equation that contains only one unknown is equal to the greatest exponent that occurs on the unknown. Table 7.1 shows some polynomial equations and their degree.

Equation	Degree	Common Name
$3x+7=0$	1	linear
$5x^2+3x-17=0$	2	quadratic
$5x^3+3x^2-8x+1=0$	3	cubic
$x^4+4x^3-9x^2+x-7=0$	4	quartic
$7x^5-3x^4+7x^3-2x^2-5x+17=0$	5	quintic

TABLE 7.1. Polynomial equations in one unknown

Definition 7.1 (Quadratic equation)
A **quadratic equation** in one unknown is any equation that can be written in the form

$$ax^2+bx+c=0,$$

where a,b and c are any numbers and $a \neq 0$. ■

A number that makes a quadratic equation true is called a "solution" or, more often, a "root" of the equation. For example, the numbers $x=1$ and $x=2$ each make the equation $x^2-3x+2=0$ true. So we say 1 and 2 are roots of $x^2-3x+2=0$.

There are several methods of solving a quadratic equation. The three we consider here are called

(1) factorization
(2) completing the square
(3) quadratic formula.

Each of these methods is based on Theorem 7.1.

Theorem 7.1
Let a and b represent any numbers. If $ab = 0$ then $a = 0$ or $b = 0$.

Proof. Let a and b represent any numbers. Suppose $ab = 0$. If a and b both equal 0, we are done. Otherwise, suppose, with no loss of generality, that $a \neq 0$. Since $a \neq 0$ we may divide by a,

$$ab = 0 \implies \frac{ab}{a} = 0 \implies b = 0.$$

■

7.1. Factorization

Several examples follow illustrate using Theorem 7.1 to solve quadratic equations by factoring.

Example 7.1

Solve $x^2 + 5x + 6 = 0$.

Solution

$$x^2 + 5x + 6 = 0$$

$$(x+2)(x+3) = 0.$$

Using Theorem 7.1,

$$x + 2 = 0 \quad \text{or} \quad x + 3 = 0.$$

$$\therefore\ x = -2 \quad \text{or} \quad x = -3.$$

Example 7.2

Solve $x^2 + 2x - 15 = 0$.

Solution

$$x^2 + 2x - 15 = 0$$

$$(x-3)(x+5) = 0.$$

Using Theorem 7.1,

$$x - 3 = 0 \quad \text{or} \quad x + 5 = 0.$$

$$\therefore\ x = 3 \quad \text{or} \quad x = -5.$$

An equation provides the opportunity to multiply both sides by a number. Example 7.3 we seize opportunity to make the coefficient of x^2 vanish.

Example 7.3

Solve $12x^2 + 108x + 216 = 0$.

Solution

There is a common factor of 12. We divide both sides by it.

$$\frac{1}{12}\left(12x^2 + 108x + 216\right) = \left(\frac{1}{12}\right)0$$

$$x^2 + 9x + 18 = 0$$

$$(x+3)(x+6) = 0.$$

Using Theorem 7.1,

$$x + 3 = 0 \quad \text{or} \quad x + 6 = 0.$$

$$\therefore\ x = -3 \quad \text{or} \quad x = -6.$$

Example 7.4

Solve $x^2 + 8x = 0$.

Solution

There is a common factor of x that we factor out first.

$$x^2 + 8x = 0$$

$$x(x+8) = 0.$$

Using Theorem 7.1,

$$x = 0 \quad \text{or} \quad x + 8 = 0.$$

$$\therefore\ x = 0 \quad \text{or} \quad x = -8.$$

Example 7.5

Solve $x^2 - 25 = 0$.

Solution

$$x^2 - 25 = 0$$

$$(x-5)(x+5) = 0.$$

Then,

$$x - 5 = 0 \quad \text{or} \quad x + 5 = 0.$$

$$\therefore\ x = 5 \quad \text{or} \quad x = -5.$$

Example 7.6

Solve $x^2 + 12x + 36 = 0$.

Solution

$$x^2 + 12x + 36 = 0. = 0$$

$$(x+6)^2 = 0.$$

Then,

$$x + 6 = 0$$

$$\therefore\ x = -6. \quad \blacksquare$$

The equations in Examples 7.1 to 7.5 on pages 144–146 each had two distinct roots. The equation in Example 7.6 had “one root multiplicity two”.

Example 7.7

Solve $-8x = -x^2 - 12$.

Solution

The equation is not yet in the form required to apply Theorem 7.1 on page 144. We begin by writing the equation with 0 on one side by itself.

$$-8x = -x^2 - 12$$

$$x^2 - 8x + 12 = 0$$

$$(x-2)(x-6) = 0.$$

Then,

$$x - 2 = 0 \quad \text{or} \quad x - 6 = 0.$$

$$\therefore\ x = 2 \quad \text{or} \quad x = 6.$$

Example 7.8

Solve $-x^2 - 5x - 6 = 0$.

Solution

$$-x^2 - 5x - 6 = 0.$$

Factoring is easier when the leading term is positive. Multiply by -1.

$$x^2 + 5x + 6 = 0$$

$$(x+2)(x+3) = 0.$$

Then,

$$x + 2 = 0 \quad \text{or} \quad x + 3 = 0.$$

$$\therefore\ x = -2 \quad \text{or} \quad x = -3.$$

Example 7.9

Solve. $100x^2 - 900x + 1400 = 0$.

Solution

$$100x^2 - 900x + 1400 = 0$$

$$x^2 - 9x + 14 = 0$$

$$(x-2)(x-7)=0.$$

Then,

$$x-2=0 \quad \text{or} \quad x-7=0.$$

$$\therefore\ x=2 \quad \text{or} \quad x=7.$$

The next examples result in rational or in irrational roots.

Example 7.10 (Rational roots)

Find all solutions of $2x^2-7x+3=0$.

Solution

$$2x^2-7x+3=0$$

$$(2x-1)(x-3)=0.$$

Then,

$$2x-1=0 \quad \text{or} \quad x-3=0.$$

$$\therefore\ x=\frac{1}{2} \quad \text{or} \quad x=3.$$

Example 7.11 (Irrational roots)

Find all roots of $x^2=7$.

Solution

$$x^2=7$$

$$x^2-7=0$$

$$(x+\sqrt{7})(x-\sqrt{7})=0.$$

Then,

$$x+\sqrt{7}=0 \quad \text{or} \quad x-\sqrt{7}=0.$$

$$\therefore\ x=-\sqrt{7} \quad \text{or} \quad x=\sqrt{7}.$$

Exercise 7.1

Solve each of the following for the unknown.

1. $b^2-b=0$
2. $n^2+10n+21=0$
3. $x^2-x-2=0$
4. $x^2+6x-16=0$
5. $a^2+4a-12=0$
6. $v^2-7v+6=0$
7. $b^2-b-42=0$
8. $5x^2-20x-60=0$
9. $m^2-10m+25=0$
10. $-x^2+5x+24=0$
11. $6x^2-12x-48=0$
12. $x^2+4x-21=0$
13. $b^2-13b+42=0$
14. $n^2+2n-35=0$
15. $m^2-11m+30=0$
16. $k^2+9k+18=0$
17. $n^2+6n-16=0$
18. $-r^2+10r-16=0$
19. $x^2-11=-x-5$
20. $6x^2-4+7x=5x^2+7x$
21. $v^2-7=-6v$
22. $5r^2+3r-19=2+4r^2+7r$
23. $b^2+2b+1=4$
24. $p^2+4p-24=6p$
25. $x^2-21=-4x$
26. $n^2+30=11n$
27. $k^2+19k+31=-4+7k$
28. $-6a^2-36+5a=5a-7a^2$
29. $9x^2-9x+14=8x^2$
30. $m^2-8m=-12$
31. $5n^2-21n+8=-8+3n$
32. $14v^2-33v-61=-5$
33. $3x^2+24x+54=-2x+6$
34. $2b^2-11b-42=-b^2$
35. $15x^2-75x+32=-7x-6x^2$
36. $7x^2-7x=x$
37. $7n^2-48n+34=-2$
38. $11a^2+4a-3=4a^2$
39. $7p^2+32p+21=5$
40. $5k^2+9k+6=-2k$
41. $35m^2+20m+4=-7m$
42. $4n^2+10n+7=n^2$

7.1.1. Equations quadratic in form

Some polynomial equations of degree greater than 2 are quadratic in form. You have already practiced factoring polynomials that are quadratic in form. So, solving equations of higher degree that are quadratic in form by factoring will seem natural to you. Before we do this, however, we must extend Theorem 7.1 on page 144 to cover any number of factors.

Theorem 7.2
Let $a_1, a_2, \cdots, a_n$ represent n-number of factors. If $a_1 \cdot a_2 \cdot \ldots \cdot a_n = 0$ then $a_1 = 0$ or $a_2 = 0$ or $\cdots$ or $a_n = 0$.

Example 7.12

Find all solutions of $x^4 - 16 = 0$.

Solution.

$$x^4 - 16 = 0$$

$$(x^2 - 4)(x^2 + 4) = 0.$$

$$(x-2)(x+2)(x^2+4) = 0.$$

Then, by Theorem 7.2,

$$x - 2 = 0 \quad \text{or} \quad x + 2 = 0 \quad \text{or} \quad x^2 + 4 = 0.$$

So,

$$x = 2 \quad \text{or} \quad x = -2 \quad \text{or} \quad \{\,\} \quad \text{(No real number squared equals } -4\text{)}.$$

The equation has two roots in the real numbers.

$$x = -2 \quad \text{or} \quad x = 2.$$

Example 7.13

Find all roots of $(x+2)^2 + 5(x+2) + 6 = 0$.

Solution.

$$(x+2)^2 + 5(x+2) + 6 = 0$$

$$((x+2)+2)((x+2)+3) = 0.$$

$$(x+4)(x+5) = 0.$$

Then,

$$x+4 = 0 \quad \text{or} \quad x+5 = 0.$$

$$\therefore\ x = -4 \quad \text{or} \quad x = -5.$$

Exercise 7.2

Solve.

1. $x^4 - 81 = 0$
2. $x^4 = 625$
3. $(x-7)^2 - (x-7) - 2 = 0$
4. $(x-7)^2 - (x-3)^2 = 0$
5. $(x-1)^2 - (x-1) = 0$
6. $(x+1)^2 + 4(x+1) + 4 = 0$
7. $(2x-1)^2 - 4(2x-1) + 4 = 0$
8. $(3x-2)^2 - (x+1)^2 = 0$
9. $(x+5)^2 - 16 = 0$
10. $(x+4)^2 - 6(x+4) + 9 = 0$

7.2. Completing the square

Our success solving quadratic equations has depended on our discovering the factorization of a polynomial. But, some polynomials are inconvenient to factor. Others are just plain hard. For example, the equation

$$x^2 - 5x + 6 = 0 \tag{7.1}$$

is easy to solve, because

$$x^2 - 5x + 6$$

is simple to factor.

$$x^2 - 5x + 6 = 0$$

$$(x-3)(x-2) = 0$$

Therefore, $x = 3$ or $x = 2$.

On the other hand, merely changing the last term of Equation 7.1 from 6 to 7 produces

$$x^2 - 5x - 7 = 0 \tag{7.2}$$

which factors as

$$\left(x - \frac{5-\sqrt{53}}{2}\right)\left(x - \frac{5+\sqrt{53}}{2}\right) = 0.$$

Hardly anyone's idea of a good time. Despair not. The procedure called "completing the square" makes solving an equation such as Equation 7.2 a routine matter.

7.2.1. Producing a square trinomial

The point of this subsection is to practice one of the several steps of completing the square.

The trinomial $x^2 + 6x + 9$ is a square trinomial. Notice that the constant term 9 is (one-half of 6) squared. Another square trinomial is $x^2 + 12x + 36$. The constant 36 is $(\frac{1}{2} \cdot 12)^2$.

What we observe in these two examples is true for *all* square trinomials whose leading coefficient is 1. That is

$$x^2 + bx + \left(\frac{b}{2}\right)^2.$$

This is no surprise when we think of producing a square trinomial:

$$(x+b)^2 = x^2 + 2bx + b^2.$$

The constant term is $(\text{one-half of } 2b)^2$ regardless of the numbers a and b..

Example 7.14

For each of the following, supply the constant term that results in a square trinomial. Then write the trinomial in its factored form.

(1) $x^2 + 8x + \_\_\_\_$

(2) $x^2 + 10x + \_\_\_\_$

(3) $x^2 - 6x + \_\_\_\_$

(4) $x^2 - 8x + \_\_\_\_$

(5) $x^2 - 12x + \_\_\_\_$

(6) $x^2 + 5x + \_\_\_\_$

(7) $x^2 - x + \_\_\_\_$

(8) $x^2 - \frac{x}{2} + \_\_\_\_$

Solution.

(1) $x^2+8x+\mathbf{16}=(x+4)^2$

(2) $x^2+10x+\mathbf{25}=(x+5)^2$

(3) $x^2-6x+\mathbf{9}=(x-3)^2$

(4) $x^2-8x+\mathbf{16}=(x-4)^2$

(5) $x^2-12x+\mathbf{36}=(x-6)^2$

(6) $x^2+5x+\mathbf{\frac{25}{4}}=\left(x+\frac{5}{2}\right)^2$

(7) $x^2-x+\mathbf{\frac{1}{4}}=\left(x-\frac{1}{2}\right)^2$

(8) $x^2-\frac{x}{2}+\mathbf{\frac{1}{16}}=\left(x-\frac{1}{4}\right)^2$

Exercise 7.3

For each of the following, supply the constant term that results in a square trinomial. Then write the trinomial in its factored form.

1. x^2+14x+ ____
2. x^2-20x+ ____
3. x^2-16x+ ____
4. x^2+7x+ ____
5. x^2-9x+ ____
6. $x^2+\frac{2}{3}x+$ ____
7. x^2+5x+
8. x^2-3x+ ____
9. x^2-6x+ ____
10. x^2-8x+ ____
11. $x^2+\frac{3}{5}x+$ ____
12. $x^2+\frac{2}{5}x+$ ____
13. $x^2-\frac{3}{7}x+$ ____
14. $x^2-\frac{1}{4}x+$ ____
15. $x^2-\frac{1}{3}x+$ ____
16. $x^2+\frac{3}{2}x+$ ____

7.2.2. Factor by completing the square, leading coefficient 1

Once we know how to select the constant term to create a square trinomial as in Example 7.14 on the facing page and Exercise 7.3, completing the square is not hard.

Before we tackle a seemingly hopeless case, we will try completing the square on several expressions whose factorization we already know. We know $x^2-6x-16=(x-8)(x+2)$.

Example 7.15

Factor $x^2 - 6x - 16$ by completing the square.

Solution

One-half of -6 is -3 and $(-3)^2 = 9$, so $x^2 - 6x + 9$ is a square trinomial. We will add and subtract 9.

$$x^2 - 6x - 16 = x^2 - 6x + 9 - 9 - 16$$

The square trinomial $x^2 - 6x + 9$ factors into $(x-3)^2$. Also, $-9 - 16 = -25$.

$$= (x-3)^2 - 25$$

This is the difference of squares. So,

$$= ((x-3)-5)((x-3)+5)$$

$$= (x-8)(x+2),$$

as expected.

Example 7.16

Factor $x^2 + 7x + 12$ by completing the square.

Solution

One-half of 7 is $\frac{7}{2}$ and $\left(\frac{7}{2}\right)^2 = \frac{49}{4}$, so $x^2 + 7x + \frac{49}{4}$ is a square trinomial. We will add and subtract $\frac{49}{4}$.

$$x^2 + 7x + 12 = x^2 + 7x + \frac{49}{4} - \frac{49}{4} + 12$$

$$= \left(x + \frac{7}{2}\right)^2 - \frac{49}{4} + 12$$

$$= \left(x + \frac{7}{2}\right)^2 - \frac{1}{4}$$

$$= \left(\left(x+\frac{7}{2}\right)-\sqrt{\frac{1}{4}}\right)\left(\left(x+\frac{7}{2}\right)+\sqrt{\frac{1}{4}}\right)$$

$$= \left(\left(x+\frac{7}{2}\right)-\frac{1}{2}\right)\left(\left(x+\frac{7}{2}\right)+\frac{1}{2}\right)$$

$$= (x+3)(x+4).$$

Example 7.17

Factor x^2-3x-1.

Solution

One-half of -3 is $\frac{-3}{2}$ and $\left(\frac{-3}{2}\right)^2=\frac{9}{4}$, so $x^2-3x+\frac{9}{4}$ is a square trinomial. We will add and subtract $\frac{9}{4}$.

$$\begin{aligned} x^2-3x-1 &= x^2-3x+\frac{9}{4}-\frac{9}{4}-1 \\ &= \left(x-\frac{3}{2}\right)^2-\frac{5}{4} \\ &= \left(\left(x-\frac{3}{2}\right)-\sqrt{\frac{5}{4}}\right)\left(\left(x-\frac{3}{2}\right)+\sqrt{\frac{5}{4}}\right) \\ &= \left(x-\frac{3}{2}-\frac{\sqrt{5}}{2}\right)\left(x-\frac{3}{2}+\frac{\sqrt{5}}{2}\right) \\ &= \left(x-\frac{3+\sqrt{5}}{2}\right)\left(x-\frac{3-\sqrt{5}}{2}\right). \end{aligned}$$

■

Example 7.17 was our first that would be difficult without completing the square.

We got started on completing the square when we considered solving Equation 7.2 on page 152. We solve this equation in Example 7.18.

Example 7.18

Solve $x^2-5x-7=0$.

Solution.

$$x^2-5x-7=0$$

$$x^2-5x+\frac{25}{4}-\frac{25}{4}-7=0$$

$$\left(x-\frac{5}{2}\right)^2-\frac{53}{4}=0$$

$$\left(x-\frac{5}{2}-\frac{\sqrt{53}}{2}\right)\left(x-\frac{5}{2}+\frac{\sqrt{53}}{2}\right)=0$$

$$\left(x-\frac{5+\sqrt{53}}{2}\right)\left(x-\frac{5-\sqrt{53}}{2}\right)=0$$

$$x-\frac{5+\sqrt{53}}{2}=0 \quad \text{or} \quad x-\frac{5-\sqrt{53}}{2}=0$$

$$\therefore\ x=\frac{5+\sqrt{53}}{2} \quad \text{or} \quad x=\frac{5-\sqrt{53}}{2}.$$

The reader has been checking solutions even though we have not been checking them in the text. This time, we will check one of the roots in print.

Check $x=\dfrac{5+\sqrt{53}}{2}$.

$$\begin{aligned}
\text{LHS} &= x^2-5x-7\\
&= \left(\frac{5+\sqrt{53}}{2}\right)^2-5\left(\frac{5+\sqrt{53}}{2}\right)-7\\
&= \frac{25+10\sqrt{53}+53}{4}-\frac{25+5\sqrt{53}}{2}-7\\
&= \frac{25+10\sqrt{53}+53}{4}-\frac{50+10\sqrt{53}}{4}-7\\
&= \frac{-25+53}{4}-7
\end{aligned}$$

$$= \frac{28}{4} - 7$$

$$= 0$$

$$= \text{RHS}.$$

■

If you think the check takes more effort than the solution, the author agrees with you.

Example 7.19

Factor $x^2 - 4x + 6$.

$$\begin{aligned} x^2 - 4x + 6 &= x^2 - 4x + 4 - 4 + 6 \\ &= (x-2)^2 + 2 \\ &= (x-2)^2 + (\sqrt{2})^2 \end{aligned}$$

The sum of squares does not factor in the real numbers. The expression in Example 7.19 cannot be factored.

7.2.3. Factor by completing the square, leading coefficient not 1

An example should make clear the process of factoring by completing the square when the leading coefficient is not 1.

Example 7.20

Factor by completing the square. $3x^2 + 5x - 4$.

Solution.

$$\begin{aligned} 3x^2 + 5x - 4 &= 3\left(x^2 + \frac{5}{3} - \frac{4}{3}\right) \qquad (7.3) \\ &= 3\left(x^2 + \frac{5}{3}x + \frac{25}{36} - \frac{25}{36} - \frac{4}{3}\right) \\ &= 3\left(\left(x + \frac{5}{6}\right)^2 - \frac{73}{36}\right) \end{aligned}$$

$$= 3\left(x + \frac{5}{6} - \sqrt{\frac{73}{36}}\right)\left(x + \frac{5}{6} + \sqrt{\frac{73}{36}}\right)$$

$$= 3\left(x + \frac{5-\sqrt{73}}{6}\right)\left(x + \frac{5+\sqrt{73}}{6}\right).$$

Equation 7.3 is the key because it makes the leading coefficient within the parenthesis is 1.

Exercise 7.4

Factor by completing the square

1. $n^2 - 4n + 1$
2. $m^2 + 4m + 1$
3. $n^2 - 2n - 1$
4. $p^2 + 6p + 1$
5. $x^2 - 8x - 9$
6. $m^2 + 5m + 5$
7. $n^2 - 7n - 19$
8. $a^2 + 3a - 7$
9. $x^2 - x - 17$
10. $x^2 + 5x - 2$
11. $2p^2 + 8p - 24$
12. $4x^2 - 8x + 3$
13. $5k^2 + 10k - 2$
14. $5a^2 + 10a - 9$
15. $3b^2 - 6b - 12$
16. $3m^2 - 10m - 8$
17. $3n^2 - 4n - 9$
18. $2m^2 - 2m - 21$
19. $3r^2 - 4r - 20$
20. $5b^2 + 9b + 2$
21. $4x^2 + 10x + 4$
22. $5k^2 + 6k - 9$
23. $2m^2 + 5m - 18$
24. $b^2 - b - 10$
25. $3n^2 - 10n - 9$

7.2.4. Solving verses factoring

An equation, unlike an expression, affords the opportunity to add (subtract) a number from both sides or to multiply (divide) both sides by a number.

Taking advantage of this when solving a quadratic equation sometimes results in work that is simpler to perform and to read. Even though the work in the examples that follow will appear different in than in the previous examples, the method is the same: factor to produce $a \cdot b = 0$, then haul out Theorem 7.1 on page 144.

Example 7.21

Solve the equation $3x^2 + 5x - 8 = 0$.

Solution

$$3x^2 + 5x - 8 = 0$$

Separate terms with the unknown and terms without the unknown,

$$3x^2 + 5x = 8.$$

Divide both sides by 3 to obtain a leading coefficient of 1,

$$x^2 + \frac{5}{3}x = \frac{8}{3}.$$

The LHS is set up for completing the square.

$$x^2 + \frac{5}{3}x + \frac{25}{36} = \frac{8}{3} + \frac{25}{36}$$

$$\left(x + \frac{5}{6}\right)^2 = \frac{121}{36}$$

$$\left(x + \frac{5}{6}\right)^2 - \frac{121}{36} = 0$$

$$\left(x + \frac{5}{6} + \sqrt{\frac{121}{36}}\right)\left(x + \frac{5}{6} - \sqrt{\frac{121}{36}}\right) = 0$$

$$\left(x + \frac{5}{6} + \frac{11}{6}\right)\left(x + \frac{5}{6} - \frac{11}{6}\right) = 0.$$

$$\left(x + \frac{16}{6}\right)\left(x - \frac{6}{6}\right) = 0.$$

$$\left(x+\frac{8}{3}\right)\left(x-1\right)=0.$$

Then, by Theorem 7.1,

$$x+\frac{8}{3}=0 \quad \text{or} \quad x-1=0.$$

$$\therefore\ x=-\frac{8}{3} \quad \text{or} \quad x=1. \qquad \blacksquare$$

Theorem 7.3 will make the work simpler. Before stating and proving Theorem 7.3, there is some handy notation to explain. We may write "$a \pm b$" instead of "$a+b$ or $a-b$". If we write "$a=\pm 5$", we mean "$a=5$ or $a=-5$". The symbol "$\pm$" is pronounced "plus or minus".

Theorem 7.3
If $a^2=b$, then $a=\pm\sqrt{b}$, provided $b \geq 0$.

Proof. Let a represent any number and b represent any nonnegative number. Suppose $a^2=b$, then

$$a^2=b \implies a^2-b=0$$

$$\implies (a-\sqrt{b})(a+\sqrt{b})=0$$

Then, by Theorem 7.1 on page 144,

$$a-\sqrt{b}=0 \quad \text{or} \quad a+\sqrt{b}=0.$$

$$\therefore\ a=\sqrt{b} \quad \text{or} \quad a=-\sqrt{b}.$$

Therefore, if $a^2=b, b \geq 0$, then $a=\pm\sqrt{b}$. $\blacksquare$

Several examples using Theorem 7.3 follow. Although the first would be easier by simple factoring.

Example 7.22

Solve by completing the square. $x^2-5x-14=0$.

Solution

$$x^2-5x-14=0$$

$$x^2 - 5x = 14$$

$$x^2 - 5x + \frac{25}{4} = 14 + \frac{25}{4}$$

$$\left(x - \frac{5}{2}\right)^2 = \frac{81}{4}.$$

By Theorem 7.3 on page 160,

$$x - \frac{5}{2} = \pm\sqrt{\frac{81}{4}}$$

$$x - \frac{5}{2} = \pm\frac{9}{2}$$

$$x = \frac{5}{2} \pm \frac{9}{2}$$

$$x = \frac{5}{2} + \frac{9}{2} \quad \text{or} \quad \frac{5}{2} - \frac{9}{2}.$$

$$\therefore\ x = 7 \quad \text{or} \quad x = -2.$$

Example 7.23

Solve. $x^2 + \frac{1}{4}x - \frac{1}{8} = 0$.

Solution

$$x^2 + \frac{1}{4}x - \frac{1}{8} = 0$$

$$x^2 + \frac{1}{4}x = \frac{1}{8}$$

$$x^2 + \frac{1}{4}x + \frac{1}{64} = \frac{1}{8} + \frac{1}{64}$$

$$\left(x + \frac{1}{8}\right)^2 = \frac{9}{64}.$$

By Theorem 7.3,

$$x + \frac{1}{8} = \pm\frac{3}{8}$$

$$x = -\frac{1}{8} \pm \frac{3}{8}.$$

$$\therefore\ x = -\frac{1}{2} \text{ or } x = \frac{1}{4}.$$

Example 7.24

Find the roots. $4x^2 - 8x + 1 = 0$.

Solution

To set the stage for completing the square, we isolate the constant on the RHS, then divide both sides by 4 to obtain a leading coefficient of 1.

$$4x^2 - 8x + 1 = 0$$

$$x^2 - 2x = -\frac{1}{4}$$

$$x^2 - 2x + 1 = -\frac{1}{4} + 1$$

$$(x-1)^2 = \frac{3}{4}.$$

By Theorem 7.3,

$$x = 1 \pm \sqrt{\frac{3}{4}}$$

$$x = 1 \pm \frac{\sqrt{3}}{2}$$

$$\therefore\ x = \frac{2 \pm \sqrt{3}}{2}.$$

Example 7.25

Solve. $4x + 5 = 3x^2$.

Solution

$$4x + 5 = 3x^2$$

$$3x^2 - 4x = 5$$

$$x^2 - \frac{4}{3} = \frac{5}{3}$$

$$x^2 - \frac{4}{3} + \frac{4}{9} = \frac{5}{3} + \frac{4}{9}$$

$$\left(x - \frac{2}{3}\right)^2 = \frac{19}{9}.$$

By Theorem 7.3,

$$x - \frac{2}{3} = \pm\sqrt{\frac{19}{9}}$$

$$x = \frac{2}{3} \pm \frac{\sqrt{19}}{3}$$

$$\therefore\ x = \frac{2 \pm \sqrt{19}}{3}.$$

Example 7.26

Solve. $4x^2 - 8x + 5 = 0$.

Solution

$$4x^2 - 8x + 5 = 0$$

$$4x^2 - 8x = -5$$

$$x^2 - 2x = \frac{-5}{4}$$

$$x^2 - 2x + 1 = \frac{-5}{4} + 1$$

$$(x - 1)^2 = \frac{-1}{4}.$$

By Theorem 7.3,

$$x - 1 = \pm\sqrt{\frac{-1}{4}}.$$

Therefore, no solution in the real numbers.

Example 7.27

Solve. $x^2 - 6x + 7 = 0$.

Solution

$$x^2 - 6x + 7 = 0$$

$$x^2 - 6x = -7$$

$$x^2 - 6x + 9 = 2$$

$$(x-3)^2 = 2.$$

$$x - 3 = \pm\sqrt{2}.$$

$$x = 3 \pm \sqrt{2}$$

In examples 7.22 to 7.26 on pages 160–163, we noted each application of Theorem 7.3. In Example 7.27 we did not. That is alright.

Example 7.28 shows what is *not* alright.

Example 7.28

Solve. $x^2 - 6x + 7 = 0$.

Solution

$$x^2 - 6x + 7 = 0$$

$$\vdots$$

(7.4) $$(x-3)^2 = 2$$

Take the square root of both sides,

(7.5) $$x - 3 = \pm\sqrt{2}.$$

But $\sqrt{2} \neq -\sqrt{2}$. Taking the square root of both sides only produces one of the two solutions. Equation 7.4 does imply Equation 7.5, but owing to Theorem 7.3.

Exercise 7.5

Solve by completing the square.

1. $b^2+7b-12=5$
2. $x^2+x-22=-3$
3. $n^2+9n+16=4$
4. $b^2-5b-2=5$
5. $b^2+3b-12=-5$
6. $p^2-p-26=-3$
7. $4x^2+3x-6=-4$
8. $2x^2+3x+1=2x$
9. $4x^2-7x-19=-4$
10. $5n^2+6n-17=-5$
11. $3b^2+b-11=4$
12. $2n^2-6n-1=5$
13. $3k^2+10k+5=0$
14. $3m^2-4m-14=0$
15. $3n^2+9n-9=0$
16. $4x^2-2x-11=0$
17. $5n^2+8n-24=0$
18. $4r^2+3r-3=0$
19. $10a=-2a^2+5$
20. $a^2=a+24$
21. $k^2=-14-9k$
22. $5x^2+4x=12$
23. $-23+3b=-b^2$
24. $a^2+3=-5a$
25. $4b=8-5a^2$
26. $m^2+3m=14$
27. $-10=-x-x^2$
28. $-9x=-3x^2-1$
29. $-4+2b=-4b^2$
30. $a^2-5a=11$
31. $k^2=9+5k$
32. $16+9a=-a^2$
33. $x^2+5x=25$
34. $m^2-9m=11$
35. $-8+7x=-x^2$
36. $3k=10-k^2$
37. $6=-3n-3n^2$
38. $2v^2-v=4$
39. $-9-9a=-4a^2$
40. $-24+2n=-2n^2$
41. $2n^2=8+6n$
42. $v^2-9v=9$

7.3. Quadratic formula

When solving quadratic equations by completing the square, you may have noticed that the routine always unfolds the same way. If we complete the square in general for $ax^2+bx+c=0$, the result is a formula for solving *any* quadratic equation.

7.3.1. Derivation of the quadratic formula

Every quadratic equation can be written in the form $ax^2+bx+c=0$ where $a \neq 0$. We solve this equation for x.

$$ax^2+bx+c=0$$

$$ax^2+bx=-c$$

$$x^2+\frac{b}{a}x=\frac{-c}{a}$$

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\frac{-c}{a}+\left(\frac{b}{2a}\right)^2$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{-c}{a}+\frac{b^2}{4a^2}$$

$$x=-\frac{b}{2a}\pm\sqrt{\frac{-c}{a}+\frac{b^2}{4a^2}}$$

$$x=-\frac{b}{2a}\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \tag{7.6}$$

■

Equation 7.6 is called *The Quadratic Formula*. You should know it by heart. If you derive it a few times on your own, you will know it.

Theorem 7.4 (The quadratic formula)
Let a,b,c,x represent any numbers, except that $a \neq 0$. Then the equation

$$ax^2 + bx + c = 0$$

has solutions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

These solutions are real numbers whenever $b^2 - 4ac \geq 0$. If the solutions are identical, we say the equation has "one root multiplicity two." ■

It is important to keep in mind that the quadratic formula solves an equation written in the exact form $ax^2 + bx + c = 0$. To apply the quadratic formula to the equation $x^2 - 4x + 4 = 0$ requires we think, if not write,

$$x^2 - 4x + 4 = 0 \iff x^2 + (-4)x + 4 = 0,$$

so

$$a = 1, b = -4, c = 4.$$

Our first Example is an equation that is easily solved by factoring, but we try the famous quadratic formula on it.

Example 7.29

Solve. $x^2 + 2x - 3 = 0$.

Solution

Write (or think) $a = 1,\ b = 2,\ c = -3$. Then,

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-2 \pm \sqrt{2^2 - 4(1)(-3)}}{2 \cdot 1} \\ &= \frac{-2 \pm \sqrt{4 + 12}}{2} \\ &= \frac{-2 \pm 4}{2} \\ &= -1 \pm 2 \end{aligned}$$

$$\therefore\ x = 1 \ \text{ or } \ x = -3.$$

Example 7.30

Solve. $4x^2 - 8x - 1 = 0$.

Solution

Write, or think, $a = 4,\ b = -8,\ c = -1$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{8 \pm \sqrt{(-8)^2 - 4(4)(-1)}}{2 \cdot 4}$$

$$= \frac{8 \pm \sqrt{64 + 16}}{8}$$

$$= \frac{8 \pm \sqrt{80}}{8}$$

$$= \frac{8 \pm 4\sqrt{5}}{8}$$

$$= \frac{2 \pm \sqrt{5}}{2}$$

$$\therefore\ x = \frac{2 + \sqrt{5}}{2} \ \text{ or } \ x = \frac{2 - \sqrt{5}}{2}.$$

It will usually be alright to leave the answer in the form $x = \frac{2 \pm \sqrt{5}}{2}$.

Example 7.31

Solve. $2x^2 - 2x + 3 = 0$.

Solution

Think $a = 2,\ b = -2,\ c = 3$. Then,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{(-2)^2 - 4(2)(3)}}{2 \cdot 2}$$

$$= \frac{2 \pm \sqrt{4-24}}{4}.$$

No solution, because $\sqrt{4-24}$ is the square root of a negative number. ■

It is unlikely anyone would choose the quadratic formula to solve $m^2 - 13 = 0$. But, Example 7.32 shows it can be done.

Example 7.32

Use the quadratic formula to solve $m^2 - 13 = 0$.

Solution

Think $a = 1,\ b = 0,\ c = -13.$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{0 \pm \sqrt{0^2 - 4(1)(-13)}}{2 \cdot 1}$$

$$= \frac{\pm\sqrt{52}}{2}$$

$$= \frac{\pm 2\sqrt{13}}{2}$$

$$\therefore\ x = \pm\sqrt{13}.$$

Example 7.33

Solve $-2x^2 + 4x - 1 = 0$.

Solution

One could think $a = -2,\ b = 4,\ c = -1$, but it is easier to use the quadratic formula when a is a positive number. So, first multiply both sides by -1 to get

$$2x^2 - 4x + 1 = 0.$$

Then, proceed as usual.

Exercise 7.6

Use the quadratic formula to solve the following equations.

1. $6k^2 + 4k - 14 = 0$
2. $6n^2 + 12n - 10 = 0$
3. $3n^2 + 12n - 22 = 0$
4. $v^2 + 8v - 10 = 0$
5. $2x^2 - 8x + 2 = 0$
6. $4r^2 + 10r - 1 = 0$
7. $10k^2 - 10k = 2$
8. $-3x^2 + 11x = -70$
9. $3x^2 = -10x + 10$
10. $5p^2 - 12 = -6p$
11. $x^2 = -3 + 6x$
12. $10m^2 + 1 = 12m$
13. $-4x^2 - 2x + 7 = 0$
14. $5x^2 - 2x - 9 = 0$
15. $-5p^2 + p + 3 = 0$
16. $-5n^2 + 10 = 0$
17. $-5x^2 + 3x + 1 = 0$
18. $2m^2 - 2m - 2 = 0$
19. $2p^2 + 3p - 8 = 0$
20. $-5n^2 - 3n + 3 = 0$
21. $b^2 + 4b - 2 = 0$
22. $b^2 - 2b - 9 = 0$
23. $4x^2 + 4x - 1 = 0$
24. $-4p^2 - 2p + 3 = 0$
25. $5k^2 - 10 = 0$
26. $x^2 + 5x + 5 = 0$
27. $-2n^2 - 4n + 3 = 0$
28. $-m^2 - 4m + 9 = 0$
29. $-5x^2 + 4x + 3 = 0$
30. $b^2 - 3b - 8 = 0$
31. $x^2 - 7 = 0 = 0$
32. $-n^2 + 5n + 2 = 0$
33. $-x^2 + 5x - 1 = 0$
34. $b^2 + 4b - 3 = 0$
35. $-2n^2 + 5n + 2 = 0$
36. $4x^2 + 5x - 1 = 0$
37. $2n^2 - n - 8 = 0$
38. $-x^2 + 3x + 7 = 0$
39. $3x^2 - 4x - 5 = 0$
40. $-5p^2 + 4p + 10 = 0$
41. $3k^2 + 2k - 7 = 0$
42. $k^2 - 6 = 0$
43. $n^2 - n - 4 = 0$
44. $b^2 + 5b - 2 = 0$
45. $n^2 - 3n + 9 = 0$
46. $-4x^2 + 10 = 0$
47. $n^2 - 4n - 1 = 0$
48. $4a^2 + 2a - 1 = 0$
49. $-4a^2 - 2a + 4 = 0$
50. $3x^2 - 5x - 1 = 0$
51. $-5a^2 - 5a + 1 = 0$
52. $-5k^2 - 2k + 9 = 0$
53. $n^2 + n - 1 = 0$
54. $4x^2 - 10 = 0$
55. $x^2 - 5x + 3 = 0$
56. $-2a^2 + 3a + 7 = 0$
57. $-m^2 + m + 9 = 0$
58. $2x^2 + 4x - 5 = 0$

7.4. Discriminant

The quantity b^2-4ac is called the "discriminant" of the equation $ax^2+bx+c=0$. It is easy to compute. It shows the quantity and nature of the roots of a quadratic equation. There are four cases.

(1) $b^2-4ac<0$. No solutions in real numbers, because the square root of a negative number is not a real number.

(2) $b^2-4ac=0$. One rational root, multiplicity two. To see why, consider
$$x=\frac{-b\pm\sqrt{0}}{2a}=-\frac{b}{2a}.$$

(3) $b^2-4ac>0$. Two real number roots.

(4) b^2-4ac is a square number. Two rational roots. To see why, suppose $b^2-4ac=N^2$. Then,
$$x=\frac{-b\pm\sqrt{N^2}}{2a}\implies x=\frac{-b+N}{2a}\text{ or }x=\frac{-b-N}{2a}$$

Example 7.34

Predict the number and kind of solutions to

(1) $3x^2+5x+2=0$.
(2) $3x^2+4x+2+0$.
(3) $4x^2-12x+9=0$.

Solution.

(1) $b^2-4ac=25-24=1>0$. Two real number roots.
(2) $b^2-4ac=16-24=-8<0$. No real number roots.
(3) $b^2-4ac=144-144=0$. One real number root, multiplicity two.

A quadratic equation $ax^2+bx+c=0$ has solutions, call them A and B, if and only if $(x-A)(x-B)=0$. This means $ax^2+bx+c=(x-A)(x-B)$. Since $ax^2+bx+c=0$ has solutions in the real numbers if and only if $b^2-4ac\geq 0$, the expression ax^2+bx+c factors if and only if $b^2-4ac\geq 0$.

Exercise 7.7

Factor.

State the quantity of real number solutions.

1. $r^2+5r+4=0$
2. $3x^2-3x-6=0$
3. $-8x^2-10x-3=0$
4. $-8n^2-8n-2=0$
5. $-7n^2-4n+3=0=0$
6. $b^2+4b-4=0$
7. $4v^2-10v+9=0$
8. $-3n^2+8n-10=0$
9. $-3x^2-9=0$
10. $5x^2-10x+5=0$
11. $9a^2-6a+1=0$
12. $-4p^2+8p-3=0$

State whether the roots are rational, irrational solutions, or nonexistent. If rational or irrational, say how many roots.

13. $2v^2+v-2=0$
14. $-5b^2+4b=0$
15. $8a^2-9a+8=0$
16. $-x^2-10x+6=0$
17. $6n^2+9n+9=0$
18. $5k^2+8k+8=0$

State whether or not the expression factors in the real numbers.

19. $4x^2+8x+4$
20. $3p^2-6p+6$
21. $3x^2+2x-7$
22. $-8x^2+6x-4$
23. $-7x+5x-3$
24. $9r^2+5r-4$

Answer each question.

25. A classmate claims the equation $ax^2+bx+c=0$ will have real number solutions whenever the signs of a and c are opposite. Is the claim always true? If you think the claim is not always true, provide a counterexample. It you think it is always true, provide a proof.
26. Explain why if $b^2-4ac<0$ the expression ax^2+bx+c does not factor as $(Ax+B)(Cx+D)$.

7.5. Summary

You have used factorization to solve quadratic equations. You will discover that factorization plays an important role in many mathematical contexts. Solving the quadratic is merely one such context.

Factorization should be your first choice for solving a quadratic equation. When the coefficient of the leading term is 1 or a prime number, then factorization is usually convenient. When the leading coefficient and the last term each have several factors, the method of Section 5.7 is effective.

If factorization is inconvenient, then the quadratic formula is the next choice. Using it requires careful attention to arithmetic, but no special insight or inspiration.

You may wonder why we bothered with completing the square if factorization and the quadratic formula are the tools for solving quadratic equations. Completing the square will be useful to you in a broad range of circumstances. You already have seen an example of this –completing the square played the key role in obtaining the quadratic formula.

When factoring an expression, do not forget to first factor out any common factor. When solving the quadratic equation, dividing by the common factor is always recommended.

Of the various ways that one might show that a quadratic expression cannot be factored in the real numbers, computing $b^2 - 4ac$ is one of the easiest. When this quantity is negative, the quadratic equation has no real number solution.

7.6. Word problems

There are typically two solutions to a quadratic equation. Each solution must be checked to be sure it is reasonable, or even meaningful, given the facts of the problem. The next example makes the point.

Example 7.35

A rectangular frame is 2 feet longer than it is wide. If the area enclosed by the frame is 15 square feet, what are the dimensions of the frame?

Solution

Let x represent the width of the frame. Then $(x+2)$ represents the length of the frame. Since the area enclosed is 15,

$$x(x+2) = 15$$

$$x^2 + 2x - 15 = 0$$

$$(x-3)(x+5) = 0.$$

Therefore,

$$x = 3 \quad \text{or} \quad x = -5.$$

Reject the formal solution $x = -5$ feet, because negative distance is nonsense. Therefore, the width is 3 feet and the length is $3 + 2 = 5$ feet.

Exercise 7.8

Answer the following.

1. The sum of two number is 9. The sum of their squares is 41. Find the numbers.
2. The square of a number added to twice the number equals 48. Find the numbers.
3. The sum of the squares of three consecutive positive numbers is 194. Find the three numbers.
4. A rectangle is 5 inches longer than it is wide. If the area of the rectangle is 104 square inches, how wide is the rectangle?
5. A rectangular flower bed is 8 feet wide by 10 feet long. A concrete walkway of uniform width surrounds the flower bed. If the total area of the flower bed and the walkway is 195 square feet, how wide is the concrete walkway?
6. A bicycle shop paid a total of \$1400 for some number of bicycles. The shop sold all but 10 of them at \$100 more per bicycle than it paid for each bicycle. If the shop's profit was \$300, how many bicycles did the shop buy?

Chapter 8

Rational functions

8.1. Inverse proportion

Speed, s, is the ratio of distance, d, to time, t. Equation 8.1 is well known

$$s = \frac{d}{t}. \tag{8.1}$$

If the speed is constant, Equation 8.1 may be rearranged to show d as a function of time. That is

$$d = st.$$

In such a case, we know that distance is directly proportional to time. Double the time, double the distance. Quadruple the time, quadruple the distance. The constant, s, is called the "constant of proportionality".

Suppose that instead of the speed being constant, the distance is constant. An example of this might be a one mile track. Then

$$d = st \tag{8.2}$$

may be rearranged to show time as a function of speed.

$$t = \frac{d}{s}.$$

We experiment with values if s to get a sense of how the function behaves. Table 8.1 on the following page shows several pairs of values of s and t.

Notice that if s is doubled, t is halved. If s is quadrupled, t is quartered. This agrees with our experience. It we travel the same distance at twice the speed, the trip will take half as long. The function $t = {}^{1}/_{s}$ is an example of "inverse proportion". We say "t is inversely proportional to s". The constant d is the "constant of proportionality".

The distinction between Equation 8.1 and Equation 8.2 is that in Equation 8.1,

$$s = \frac{d}{t},$$

s	t
1/6	6
1/5	5
1/4	4
1/3	3
1/2	2
1	1
2	1/2
4	1/4
6	1/6
8	1/8
10	1/10

TABLE 8.1. Some values of the function $t = {}^1/_s$

the *ratio is constant*.

But in Equation 8.2,

$$d = st,$$

the *product is constant*.

8.2. The nature of y = $^1/_x$

The argument of the function $t = \frac{d}{s}$ is s and it is in the denominator. This is new to our story. Naturally, we are curious about the character of this function. We begin with the special case where the constant of proportionality d in $t = \frac{d}{s}$ is 1. We will use letters x and y for the variables in the discussion that follows. In this section, we discuss Equation 8.3.

$$y = \frac{1}{x}. \tag{8.3}$$

Our discussion of $y = {}^1/_x$ will be lengthy, so we will keep track of our discoveries by calling them "Facts" in bold typeface and numbering them. At the end of our discussion, we will round up the various facts into a concise summary.

Since x is in the denominator, we know that x cannot take the value 0. So, 0 is excluded from the domain of $y = {}^1/_x$. This means that the graph of $y =$

1/x cannot intersect the y-axis. We also note that the sign of y is determined by the sign of x, since the numerator is positive.

Fact 1: $x \neq 0$.

Fact 2: $y = 1/x$ does not intersect y-axis.

The graph of $y = 1/x$ is not a straight line, because the slope is not constant. Select the three points $P(1,1), Q(2,\frac{1}{2}), R(3,\frac{1}{3})$ on $y = 1/x$, then compute the slope, m_{PQ} using points P and Q and the slope, m_{QR} using points Q and R.

$$m_{PQ} = \frac{\frac{1}{2}-1}{2-1} = -\frac{1}{2}, \quad m_{QR} = \frac{\frac{1}{3}-\frac{1}{2}}{3-1} = -\frac{1}{6}$$

Since $m_{PQ} \neq m_{QR}$, the portion of $y = 1/x$ from $x = 1$ to $x = 3$ is not a straight line. This leaves open the question of whether or not $y = 1/x$ might have straight segments elsewhere. We will settle that question on page 181. For now we can say that the graph is not a straight line overall, because a portion of it is not straight.

Fact 3: At least a portion of y = 1/x is not a straight line.

We cannot talk about the value of $y = 1/x$ when $x = 0$, since that is undefined. But we can ask about how $y = 1/x$ behaves when x is near $x = 0$. Of course x can be to the left or to the right of 0.

We first investigate when x is near and to the right of 0. Since x is to the right of 0, x is positive and so $y = 1/x$ is positive, too. Now, imagine that while staying to the right of 0, x takes on values closer and closer to 0. In fact, you might imagine x a movable point on the x-axis that is moving closer and closer to 0 from the right.

Pause and recall that the smaller the denominator of a fraction, the greater is the fraction. For example, 1/10 is greater than 1/100 and 1/100 is greater than 1/1000.

As x takes values closer to 0 from the right, $y = 1/x$ becomes greater in the positive (up) direction. Figure 8.1 on the next page shows this.

Fact 4: As x takes values closer to 0 from the right, $y = 1/x$ grows greater in the positive direction.

Now imagine that x takes values ever closer to 0, but *from the left*. Since x is negative, $y = 1/x$ is negative too. So, $y = 1/x$ will become great, but in the negative (down) direction. Figure 8.2 on the following page.

Fact 5: As x takes values closer to 0 from the left, $y = 1/x$ grows greater in the negative direction.

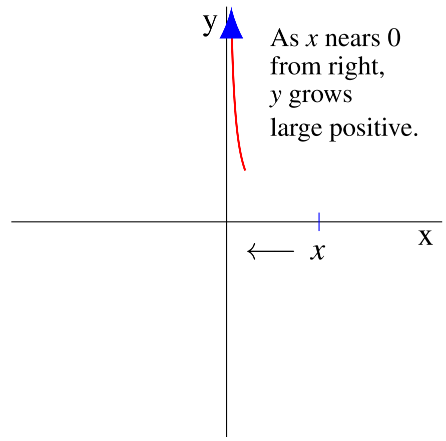


FIGURE 8.1. $y = {}^1/_{\mathrm{x}}$, x is approaches 0 from the right.

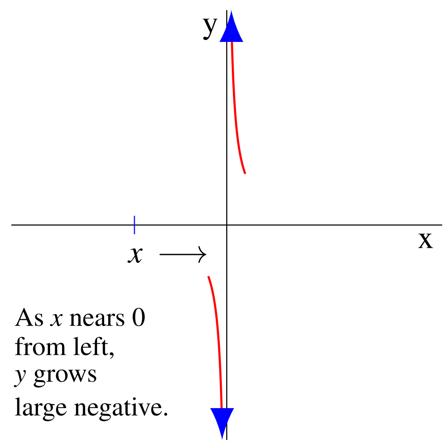


FIGURE 8.2. $y = {}^1/_{\mathrm{x}}$, x is approaches 0 from the left.

How does $y = {}^1/_{\mathrm{x}}$ behave as x takes values farther and farther to the right? Since x will be positive, $y = {}^1/_{\mathrm{x}}$ is positive too. As the denominator of $y = {}^1/_{\mathrm{x}}$ becomes large, $y = {}^1/_{\mathrm{x}}$ gets closer to 0. Figure 8.3 on the next page.

Fact 6: As x moves farther right, $y = {}^1/_{\mathrm{x}}$ approaches 0 from above.

How does $y = {}^1/_{\mathrm{x}}$ behave as x takes values farther and farther to the left? Since x will be negative, $y = {}^1/_{\mathrm{x}}$ is negative too. As the denominator of $y = {}^1/_{\mathrm{x}}$ becomes large, $y = {}^1/_{\mathrm{x}}$ gets closer to 0, but stays negative. Figure 8.4 on the facing page.

Fact 7: As x moves farther to the left, $y = {}^1/_{\mathrm{x}}$ approaches 0 from below.

We have gained some insight into $y = {}^1/_{\mathrm{x}}$. But these is a gap in our knowledge. Well, two gaps actually. They are pretty obvious in Figure 8.4 on the next page.

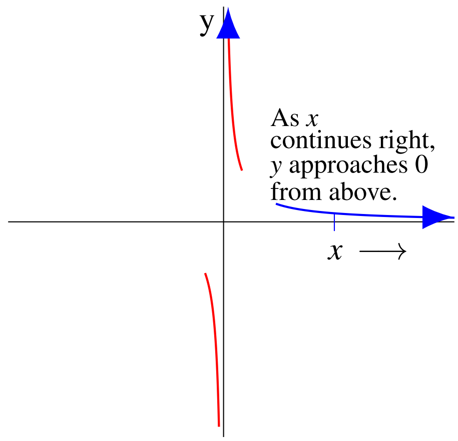


FIGURE 8.3. $y = {}^1/_X$, x takes values farther and farther right.

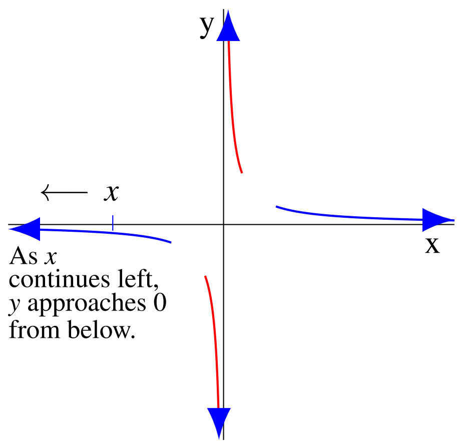


FIGURE 8.4. $y = {}^1/_X$, x takes values farther and farther to the left.

The best we can do is plot some points to get an idea of how $y = {}^1/_X$ behaves in these two regions. Then take it on faith that were we to plot more points, $y = {}^1/_X$ would behave predictably. If you find this unsatisfying, be patient. In a few years you will have more powerful mathematical tools to use.

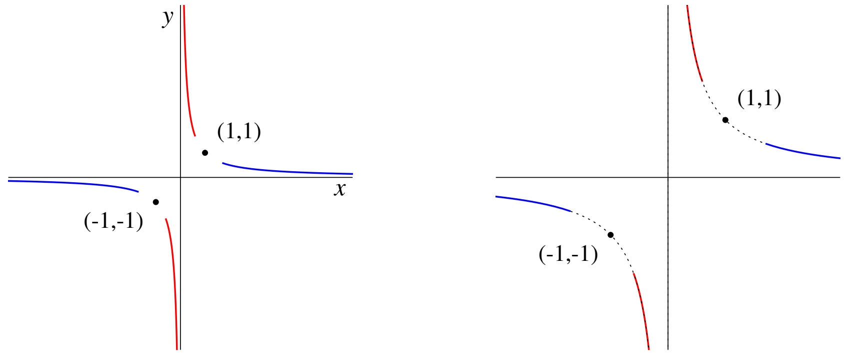
The first points to plot are $(-1,-1)$ and $(1,1)$, shown in (a) of Figure 8.5 on the following page. They are easy to compute. More points suggest the shape of the graph as is in (b) of Figure 8.5 on the next page.

Fact 8: Points $(-1,-1)$ and $(1,1)$ are on the graph of $y = {}^1/_X$.

Figure 8.6 on the following page shows the final graph.

8.2.1. Summary

The characteristics of $y = {}^1/_X$ that we have discovered are listed here:



(a) Two easily computed points plotted. (b) More points plotted.

FIGURE 8.5. Evidence (no proof) for the shape of the graph near points $(-1,-1)$ and $(1,1)$.

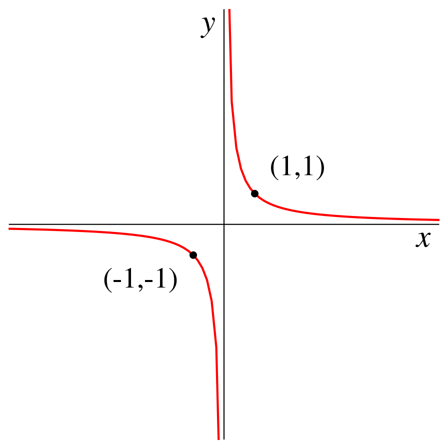


FIGURE 8.6. $y = {}^{1}/_{x}$.

(1) $x \neq 0$. Equivalently, 0 is not in the domain of $y = {}^{1}/_{x}$.
(2) $y = {}^{1}/_{x}$ does not intersect the y-axis.
(3) $y = {}^{1}/_{x}$ is not a straight line.
(4) As x approaches 0 from the right, y becomes increasingly positive; that is, the graph heads upward.
(5) As x approaches 0 from the left, y becomes increasingly negative; that is, the graph heads downward.
(6) As x takes values farther to the right, y approaches 0 through positive values (from above the x-axis).
(7) As x takes values farther to the left, y approaches 0 through negative values (from below the x-axis).
(8) Points $(-1,-1)$ and $(1,1)$ are on the graph.
(9) We expect the graph has the curved shape shown in Figure 8.6.

8.2.2. No portion of y = 1/x is straight

On page 177, we discovered that at least one stretch of the line $y = {}^1/\text{x}$ is not straight. We wondered if any portion of it is straight. We have the complete graph in Figure 8.6 on the facing page. It starts looking pretty straight far to the right and left.

Perhaps, on that never-ending line, there is some tiny portion that is straight. A straight segment that is so short, we will not see it even on a highly magnified graph. How can we be sure *that* does not occur?

In fact, *no* portion of the line $y = {}^1/\text{x}$ is straight. Here is why. Consider the branch of $y = {}^1/\text{x}$ in the first quadrant. Choose any two distinct numbers that are to the right of 0 on the x-axis and call them x_1 and x_2. Since the number line is continuous, there is a real number between x_1 and x_2, call that number x. Since $y = {}^1/\text{x}$ is defined for all $x > 0$, there are points $P(x_1, y_1), R(x, y)$ and $Q(x_2, y_2)$ on $y = {}^1/\text{x}$. Compute the slopes

$$m_{RP} = \frac{y - y_1}{x - x_1} = \frac{\frac{1}{x} - \frac{1}{x_1}}{x - x_1} = \frac{\frac{x_1 - x}{xx_1}}{x - x_1} = \left(\frac{x_1 - x}{xx_1}\right)\frac{1}{x - x_1} = \frac{-1}{xx_1},$$

and

$$m_{RQ} = \frac{y - y_2}{x - x_2} = \frac{\frac{1}{x} - \frac{1}{x_2}}{x - x_2} = \frac{\frac{x_2 - x}{xx_2}}{x - x_2} = \left(\frac{x_2 - x}{xx_2}\right)\frac{1}{x - x_2} = \frac{-1}{xx_2}.$$

Suppose that the portion of PQ of $y = {}^1/\text{x}$ is straight. Then, m_{RP} must equal m_{RQ}. So,

$$m_{RP} = m_{RQ}$$

$$\frac{-1}{xx_1} = \frac{-1}{xx_2}$$

$$x_1 = x_2. \tag{8.4}$$

But Equation 8.4 contradicts the fact that x_1 and x_2 are two distinct numbers. Therefore, no portion of $y = {}^1/\text{x}$ in the first quadrant is straight. Repeating this argument for the branch of $y = {}^1/\text{x}$ in the fourth quadrant proves that no portion of $y = {}^1/\text{x}$, no matter how short, is straight.

8.3. y = a/x . It's a whole family.

You have probably noticed that in some families the members bear a family resemblance to one another. The whole clan of $y = {}^\text{a}/\text{x}$ share a family resemblance.

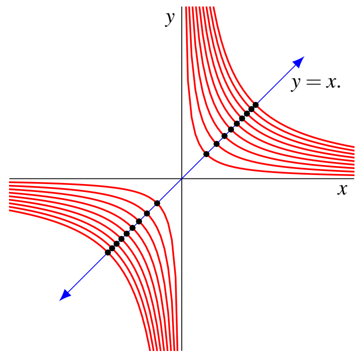


FIGURE 8.7. $y = {}^a/_x$. Family portrait.

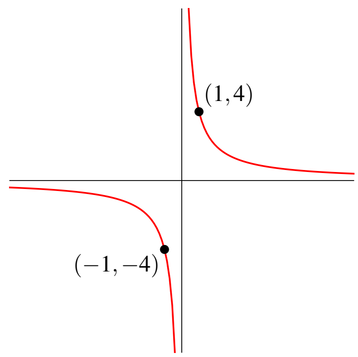


FIGURE 8.8. The function $y = {}^4/_x$ is symmetric with respect to the origin.

Various values of a produce the various family members. In Figure 8.7, the values of a are $a = 1,2,3,4,5,6,7,8,9$. The straight line shown is the line $y = x$. The curve $y = {}^a/_x$, $a \neq 0$, is called a hyperbola.

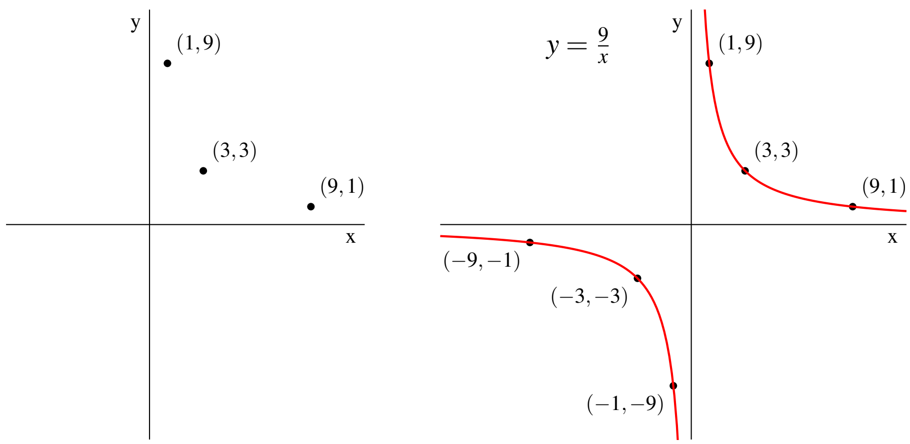
8.4. Symmetry

The function $y = \frac{a}{x}$ is symmetric with respect to the origin. This means that the point (u,v) is on $y = \frac{a}{x}$ if and only if point $(-u,-v)$ is also on $y = \frac{a}{x}$. Figure 8.8 illustrates this type of symmetry using points $(1,4)$ and $(-1,-4)$ as an example pair. If a function has symmetry, you can graph it with less effort.

Example 8.1

Graph the function $y = \frac{9}{x}$.

Solution



(a) Plot $(1,9),(9,1),(3,3)$. (b) Plot symmetric points.

FIGURE 8.9. Graph $y = \frac{9}{x}$

Points $(1,9),(9,1),(3,3)$ are on the graph. Symmetry guarantees that points $(-1,-9),(-9,-1),(-3,-3)$ are also on the graph. See Figure 8.9.

■

When producing a graph of a $y = \dfrac{a}{x}$, the graph at the extremes of the domain is easy. We just ask what happens to y as x takes values far to the right, far to the left, and closer to 0 from the right and from the left.

Slightly harder is obtaining the obviously curved portion near the origin. Three points are enough to give a pretty good idea of the shape. It is especially desirable to know the point whose coordinates are equal, because it, in a sense, "anchors" the curve.

The points $(1,y)$ and $(x,1)$ are obtained by substituting 1 for x and then 1 for y. It is obvious the points will always be $(1,a)$ and $(a,1)$.

The point whose coordinates are equal (in Example 8.1 on the facing page it was the point $(3,3)$) may be a little more trouble.

Finding the point whose coordinates are equal amounts to solving

$$u = \frac{a}{u}.$$

So, u will always equal $\sqrt{a}$, providing $a > 0$. We did not need to, but we could have obtained the point $(3,3)$ in Example 8.1 by solving

$$u = \frac{9}{u}.$$

Example 8.2

Graph the function $y = \dfrac{5}{x}$.

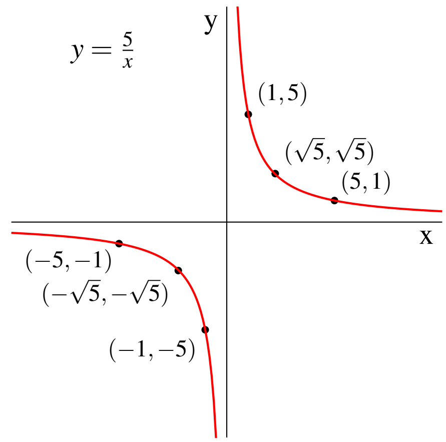


FIGURE 8.10. Graph $y = \frac{5}{x}$

Solution

The points $(1,5)$ and $(5,1)$ are easy to get by simple mental substitution. The anchor point where the coordinates are equal has coordinates u where $u = \sqrt{5}$. Although $\sqrt{5}$ is completely correct, it is inconvenient when you are trying to put a point on a graph. Graphing by hand in mathematics is an experience in imprecise drawing, so any rational approximation of $\sqrt{5}$ that is not too silly should be acceptable. The approximation $2.25 \approx \sqrt{5}$ is more than good enough. Figure 8.10 pictures what we know about $y = \dfrac{5}{x}$.

Do not be worried about approximating an irrational number when it turns up in this context. Use the method of Section 3.3.7 on page 64 or a calculator. Section 8.4.1 explains how to find a rough approximation that will usually be good enough for graphing.

8.4.1. Rough approximation of square roots

You know how to expand a binomial $(a+b)^2$. This makes finding an approximation good enough for graphing, a quick mental (well, with practice) exercise. Suppose you need a quick and rough approximation of $\sqrt{5}$. Here is what you might think.

$$4 < 5 < 9.$$

So,

$$\sqrt{4} < \sqrt{5} < \sqrt{9}$$

$$2 < \sqrt{5} < 3.$$

We chose 4 and 9 because they are consecutive square numbers that 5 lies between.

$$\sqrt{5} \approx 2 + \frac{1}{4}.$$

Then we check if this approximation is close enough for graphing.

$$\left(2 + \frac{1}{4}\right)^2 = 4 + 2(2)\left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)^2.$$

Ignore the last term, because $\left(\frac{1}{4}\right)^2$ is small compared to the number 5. The sum of the first two terms is 5. So, let's use the approximation $\sqrt{5} \approx 2.25$.

Example 8.3

Find a rough approximation to $\sqrt{18}$.

Solution

18 is closer to $16 = 4^2$ than to $25 = 5^2$, so we try 4.25.

$$\left(4 + \frac{1}{4}\right)^2 \approx 16 + 2(4)\left(\frac{1}{4}\right) = 18. \quad \blacksquare$$

But be careful. If the number whose approximate square root you seek is not close enough to a square number, the approximation might not be good enough for your purpose. Example 8.4 is such a case.

Example 8.4

Find a rough approximation to $\sqrt{20}$.

Solution

$16 < 20 < 25$ and 20 is closer to 16 than to 25. So we try 4.25.

$$\left(4 + \frac{1}{4}\right)^2 \approx 16 + 2 \cdot 4 \cdot \frac{1}{4} + \left(\frac{1}{4}\right)^2 \approx 18.$$

Since 18 is not very close to 20, this may not be a good enough approximation of $\sqrt{20}$. If you are thinking that since 20 is about midway between 16 and 25 the author should have thought to try $\sqrt{20} \approx 4.5$, you are ahead of the game.

$$\left(4 + \frac{1}{2}\right)^2 \approx 16 + 2 \cdot 4 \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^2 \approx 20.$$

Exercise 8.1

Graph each of the following

1. $y = \frac{9}{x}$
2. $y = \frac{16}{x}$
3. $y = -\frac{9}{x}$
4. $y = \frac{6}{x}$

Answer the following.

5. We found that $\sqrt{5} \approx 2.25$. Use a calculator to find $\sqrt{5}$. Compare the calculator value to 2.25.
6. In Example 8.3 on the previous page, we found that $\sqrt{18} \approx 4.25$. Use a calculator to find $\sqrt{18}$. Compare the calculator value to 4.25
7. In Example 8.4 on the preceding page, we found that $\sqrt{20} \approx 4.5$. Use a calculator to find $\sqrt{20}$. Compare the calculator value to 4.5
8. Approximate $\sqrt{68}$. Compare your result with a calculator value.
9. Approximate $\sqrt{72}$. Compare your result with a calculator value.
10. Approximate $\sqrt{109}$. Compare your result with a calculator value.

Write the function of the form $y = \frac{a}{x}$ if the point given is on its graph.

11. $(7,7)$
12. $(1,13)$
13. $(11,1)$
14. $(\sqrt{8},\sqrt{8})$
15. $(-5,1)$
16. $(-\sqrt{2},\sqrt{2})$
17. $(-6,6)$

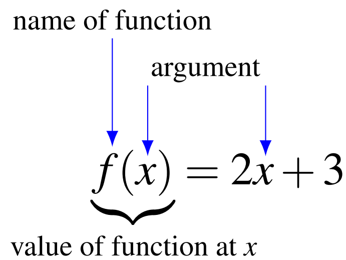


FIGURE 8.11. Parts of "$f(x) = 2x+3$" and their roles

8.5. Notation for functions

There is a widely used notation for functions. Using it in the discussion of some topics that are coming up would result in more clarity with less tortured prose. So, now is the time for it. We will explain the new notation using the familiar linear function.

We know that $y = 2x+3$ expresses y as a function of x. The expression $2x+3$ shows how to compute the value of the function, y, given an argument, x. When the argument, x, of the function is 9, the value of the function, y, is 21. When the argument is -2, the value is -1. If we like, we can name to the function $y = 2x+3$ to make referring to it easy.

Here is how we would expresses these ideas for the function $y = 2x+3$.

$$f(x) = 2x+3.$$

We pronounce $f(x) = 2x+3$ saying "f of x equals $2x+3$". But we think "f names the function whose value at argument x is $2x+3$." Figure 8.11 further the meanings of the parts of "$f(x) = 2x+3$". Popular names for functions are usually letters from the middle of the alphabet like f, g, h, j.

You will often hear people say "the function f of x." When you hear or read this, keep in mind that the function is called "f" and "f of x" indicates the value of the function at x.

Note that "$f(x)$" and "$2x+3$" each name the value of the function at x. We can refer to the value of the function f when its argument is 3 by writing "$f(3)$". The value of f at -137 is written "$f(-137)$".

If the function f is defined by $f(x) = x+4$, then

$$f(2) = 6,$$

$$f(-9) = -5,$$

$$f(\sqrt{3}) = \sqrt{3}+4,$$

$$f\left(\frac{2}{3}\right) = \frac{14}{3}.$$

When we wish to define f as the function $f(x) = 3x+7$, we write "$f : f(x) = 3x+7$".

Example 8.5

Let $f(x) = 2x+1$, $g(x) = \frac{1}{2}x$ and $h(x) = \sqrt{x}$. Find

(1) $f(3)$
(2) $f(-1)$
(3) $g(3)$
(4) $h(9)$
(5) $f(6)+h(4)$
(6) $f(2)+g(10)+h(4)$
(7) $f(10)-g(2)$
(8) $f(-2)\cdot h\left(\frac{25}{49}\right)$

Solution

(1) $f(3) = 2\cdot 3+1 = 7$
(2) $f(-1) = 2(-1)+1 = -1$
(3) $g(3) = \frac{1}{2}\cdot 3 = \frac{3}{2}$
(4) $h(9) = \sqrt{9} = 3$

(5) $f(6)+h(4) = (2\cdot 6+1)+\sqrt{4} = 13+2 = 15$
(6) $f(2)+g(10)+h(4) = (2\cdot 2+1)+(\frac{1}{2}\cdot 10)+\sqrt{4} = 5+5+2 = 12$
(7) $f(10)-g(2) = (2\cdot 10+1)-(\frac{1}{2}\cdot 2) = 21-1 = 20$
(8) $f(-2)\cdot h\left(\frac{25}{49}\right) = (2(-2)+1)\cdot\sqrt{\frac{25}{49}} = -3\cdot\frac{5}{7} = \frac{-15}{7}$

Example 8.6

Suppose a tank that initially contains 200 gallons of molasses leaks at a rate of 1.5 gallons per day. Write the volume of molasses as a function V of the number of days t the tank leaks. Then compute the volume of molasses in the tank at the end of 5 leaking days.

Solution

Let t represent the number of days leaking. Let $V(t)$ = the amount in gallons of molasses in the tank. Then

$$V(t) = 200-(1.5)t.$$

When $t = 5$, $V(t) = V(5) = 200-(1.5)(5) = 192.5$.

Exercise 8.2

Let $f(x) = \sqrt{x-1}$, $g(x) = 3x$, *and* $h(x) = 5x - 2$. *Find each of the following.*

1. $f(10)$
2. $f(37)$
3. $g(5)$
4. $g(12)$
5. $h(20)$
6. $h\left(\frac{1}{5}\right)$
7. $f(17) + g(2)$
8. $f(5) - f(10)$
9. $f(17) \div g(6)$
10. $h(4)(f(5) + g(2))$

Answer each of the following. Write the function using function notation

11. A worm farm in Detroit makes a gross profit of \$15 per 1000 Red Wigglers. Write the gross profit as a function P of x number of Red Wigglers sold. Then compute the gross profit from a sale of 20000 worms.
12. A tractor gets 10 miles per gallon of diesel fuel. Write the amount of fuel A as a function of miles x traveled. Then compute the fuel needed for a 1000 mile trip.

8.6. Translation of graphs

8.6.1. Vertical translation

Let f be the function defined by $f(x) = \frac{1}{x}$. Then suppose we add a constant k to every value of f. The result is a new function, call it g, where

$$g(x) = f(x) + k. \tag{8.5}$$

If you have guessed that the number k in Equation 8.5 shifts the graph of f vertically up by k units when $k > 0$ and shifts f vertically down by k units when $k < 0$, then you are correct. See Figure 8.12

The word “translates” is often used instead of “shift”. We say that the constant k “translates f vertically”. The word “translation” is used in mathematics and physics to refer to motion that is in a straight line.

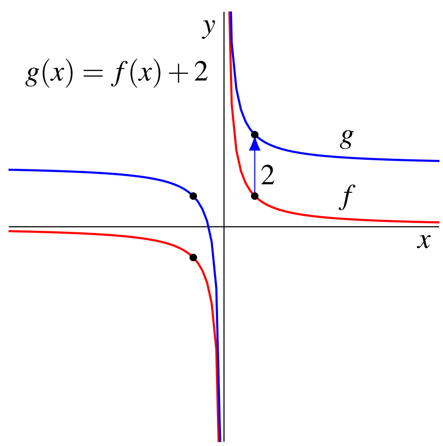


FIGURE 8.12. $y = \frac{1}{x}$ translated 2 units up.

8.6.2. Horizontal translation

Study panel (**a**) of Figure 8.13 on the next page. You see two copies of the first quadrant branch of $y = \frac{1}{x}$. Call the copy on the left “g” and the copy on the right “f”.

g is f shifted h units left.

Now we make an important observation.

> The value y paired with x by function g may be obtained from function f using argument $x + h$.

In other words,

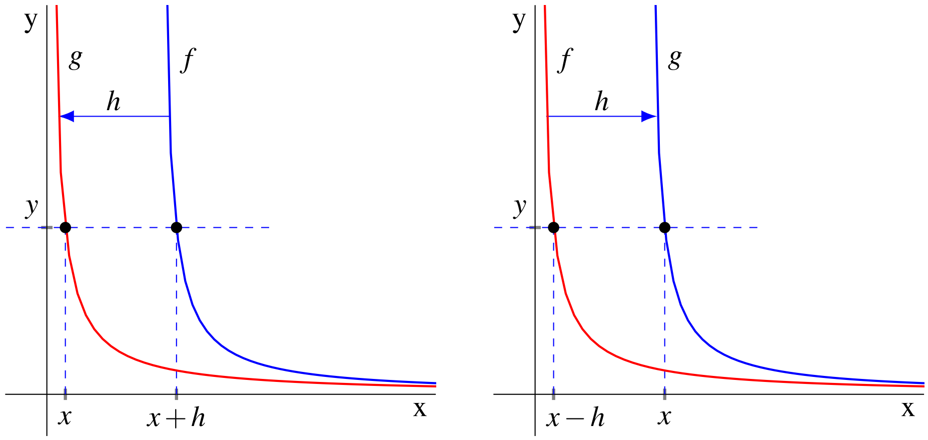


FIGURE 8.13. $y = \frac{1}{x}$ translated h units horizontally.

$g(x)$ and $f(x+h)$ are the same number.

Since $f(x+h) = \frac{1}{x+h}$,

$$g(x) = \frac{1}{x+h}.$$

Conclusion: $y = \frac{1}{x+h}$ is $y = \frac{1}{x}$ shifted h units to the left.

Now study panel (**b**) of Figure 8.13. Again there are two copies of the first quadrant branch of $y = \frac{1}{x}$. In this panel

g is f shifted h units right.

We note that

> The value y paired with x by function g may be obtained from function f using argument $x - h$.

In other words,

$g(x)$ and $f(x-h)$ are the same number.

Since $f(x-h) = \frac{1}{x-h}$,

$$g(x) = \frac{1}{x-h}.$$

Conclusion: $y = \frac{1}{x-h}$ is $y = \frac{1}{x}$ shifted h units to the right.

Both conclusions can be expressed in one statement,

Theorem 8.1

For any real numbers $x \neq 0$ and $h \neq x$, $y = \frac{1}{x-h}$ is $y = \frac{1}{x}$ shifted h units horizontally. When $h > 0$, the shift is to the right. When $h < 0$, the shift is to the left.

Example 8.7

The graph of $y = \frac{1}{x-3}$ is the graph of $\frac{1}{x}$ translated. But how far and in what direction?

Solution

Since $y = \frac{1}{x-3}$ is in the form $y = \frac{1}{x-h}$, by inspection $h > 0$. So the graph of $y = \frac{1}{x-3}$ is $\frac{1}{x}$ translated 3 units to the right.

Example 8.8

Discuss the function $g : g(x) = \frac{1}{x+5}$ by comparing it to the function $f : f(x) = \frac{a}{x}$.

Solution

Rewrite $g(x)$ in the form $f(x) = \frac{a}{x-h}$.

$$\begin{aligned} g(x) &= \frac{1}{x+5} \\ &= \frac{1}{x-(-5)}. \end{aligned}$$

By inspection, $a = 1$ and $h < 0$, so g is $f : f(x) = \frac{1}{x}$ shifted 5 units left.

Example 8.9

Discuss the function $g : g(x) = \frac{4}{x+1}$ by comparing it to the function $f : f(x) = \frac{a}{x}$.

Solution

Rewrite $g(x)$ in the form $f(x) = \frac{a}{x-h}$.

$$\begin{aligned} g(x) &= \frac{4}{x+1} \\ &= \frac{4}{x-(-1)}. \end{aligned}$$

By inspection, $a = 4$ and $h < 0$, so g is $f : f(x) = \frac{4}{x}$ shifted 1 unit left. Since the graph of $f(x) = \frac{4}{x}$ passes through the point $(2,2)$, the graph of $g(x) = \frac{4}{x+1}$ passes through $(1,2)$.

Example 8.10

Discuss the function $g : g(x) = \frac{1}{2x-6}$ by comparing it to the function $f : f(x) = \frac{a}{x}$.

Solution

We need to rewrite $g(x)$ in precisely the form $y = \frac{a}{x-h}$. Note that the coefficient of x must be 1.

$$g(x) = \frac{1}{4x-12}$$

$$= \frac{1}{4(x-3)}$$

$$= \frac{\frac{1}{4}}{x-3}$$

By inspection, $a = \frac{1}{4}$ and $h = 3$. So, g is $f(x) = \frac{0.25}{x}$ shifted 3 units right. Since the graph of $f(x) = \frac{0.25}{x}$ passes through the point $(0.5, 0.5)$, the graph of $g(x) = \frac{1}{2x-6}$ passes through $(3.5, 0.5)$.

Example 8.11

Discuss the function $g : g(x) = \frac{1}{x-7} + 3$ by comparing it to the function $f : f(x) = \frac{a}{x}$.

Solution

This is the form $f(x) = \frac{a}{x-h} + k$. By inspection, this is the graph of $f : f(x) = \frac{1}{x}$ translated 7 units right and 3 units up.

Exercise 8.3

Discuss each function by comparing it to the function f.

1. Compare $g : g(x) = \frac{1}{x-4}$ to $f : f(x) = \frac{1}{x}$.
2. Compare $g : g(x) = \frac{1}{x+6}$ to $f : f(x) = \frac{1}{x}$.
3. Compare $h : h(x) = \frac{2}{x-1}$ to $f : f(x) = \frac{2}{x}$.
4. Compare $h : h(x) = \frac{3}{x+5}$ to $f : f(x) = \frac{3}{x}$.
5. Compare $h : h(x) = \frac{1}{2x-8}$ to $f : f(x) = \frac{1/2}{x}$.
6. Compare $h : h(x) = \frac{1}{5x+15}$ to $f : f(x) = \frac{\frac{1}{5}}{x}$.
7. Compare $g : g(x) = \frac{3}{3x-6} - 5$ to $f : f(x) = \frac{1}{x}$.

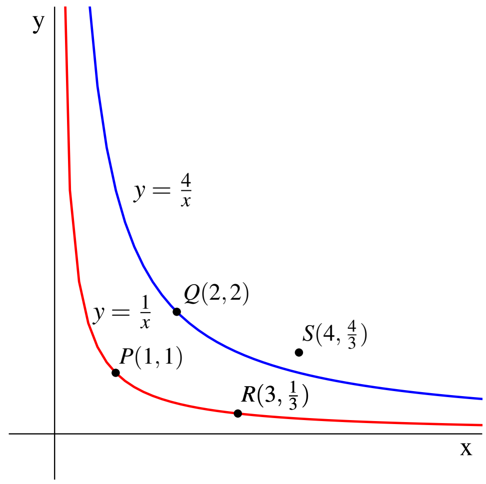


FIGURE 8.14. $y = \dfrac{4}{x}$ is not a translation of $y = \dfrac{1}{x}$.

8.6.3. Could $y = a/x$ be $y = 1/x$ translated?

Now that we have discussed translation, maybe we ought reconsider the family of functions mentioned in Section 8.3. Perhaps $y = \dfrac{a}{x}$, $a \neq 1$ is not a different member of the family but really just $y = \dfrac{1}{x}$ translated to a different location. Could, for example, $y = \dfrac{4}{x}$ just be a translation of $y = \dfrac{1}{x}$?

The answer is "No". Here is why. The point $P(1,1)$ is on $y = \dfrac{1}{x}$ and the point $Q(2,2)$ is on $y = \dfrac{4}{x}$. To move point $P(1,1)$ to $Q(2,2)$ add $h = 1$ to $x = 1$ and $k = 1$ to $y = 1$. Note that every point of $y = \dfrac{4}{x}$ should be a point of $y = \dfrac{1}{x}$ moved 1 unit right and 1 unit up. Now, $R(3, \frac{1}{3})$ is a point on $y = \dfrac{1}{x}$. Shifting point R 1 unit right and 1 unit up places R at $S(4, \dfrac{4}{3})$. But, when $x = 4$, function $y = \dfrac{4}{x}$ produces $(4,1)$ not $S(4, \dfrac{4}{3})$. We conclude that $y = \dfrac{4}{x}$ is not just $y = \dfrac{1}{x}$ at a new location. Figure 8.14 will help make the reasoning clear.

Based strictly on the appearance of $y = \dfrac{1}{x}$ and $y = \dfrac{a}{x}$, this result might seem a little surprising.

Visually,

$$y = \frac{a}{x}$$

appears more like

$$y = \frac{1}{x}$$

than it does to

$$y = \frac{1}{x-h} + k.$$

But, you know that when $a \neq 1$,

$$y = \frac{a}{x} \quad \text{and} \quad y = \frac{1}{x}$$

are different members of a family, whereas

$$y = \frac{1}{x} \quad \text{and} \quad y = \frac{1}{x-h} + k$$

are really the same member, but merely observed at different locations in the coordinate plane.

8.7. Asymptotes

The function $y = 1/x$ approaches the x-axis and the y-axis getting closer and closer to each. This behavior makes each axis an *asymptote* of the $y = 1/x$. The y-axis is a vertical asymptote of the function and the x-axis is a horizontal asymptote of $y = 1/x$. When we graph a function that has an asymptote, we usually show the asymptote as a dashed line. But when an axis is an asymptote, we leave it as a solid line.

Example 8.12

Graph the function $y = \dfrac{1}{x-4} + 3$.

Solution

The graph is that of $y = \dfrac{1}{x}$ translated 4 units to the right and 3 units up. The points and $(1,1), (2,1/2)$ and $(1/2,2)$ are on $y = \dfrac{1}{x}$. By symmetry, the points $(-1,-1), (-2,-1/2), (-1/2,-2)$ are also on $y = \dfrac{1}{x}$. Table 8.2 shows the point on $y = \dfrac{1}{x}$ and the location to which the point shifts. The last two lines of Table 8.2 shows the translation of asymptotes. The graph appears in Figure 8.15 on the facing page.

8.7.1. A slightly alternative approach

Another way to think of the translation of $y = \dfrac{a}{x}$ is to imagine the curve $y = \dfrac{a}{x}$ *and* the coordinate system in which it lives shifted h units horizontally and k units vertically. The asymptotes of $y = \dfrac{a}{x}$ are $x = 0$ and $y = 0$. They

point $(-2,-1/2)$	shifts to	$(2,2.5)$
point $(-1,-1)$	shifts to	$(3,2)$
point $(-1/2,-2)$	shifts to	$(3.5,1)$
point $(1/2,2)$	shifts to	$(4.5,5)$
point $(1,1)$	shifts to	$(5,4)$
point $(2,1/2)$	shifts to	$(6,3.5)$
asymptote $x=0$	shifts to	$x=4$
asymptote $y=0$	shifts to	$y=3$

TABLE 8.2. Translation of $y=\frac{1}{x}$ by $h=4, k=3$.

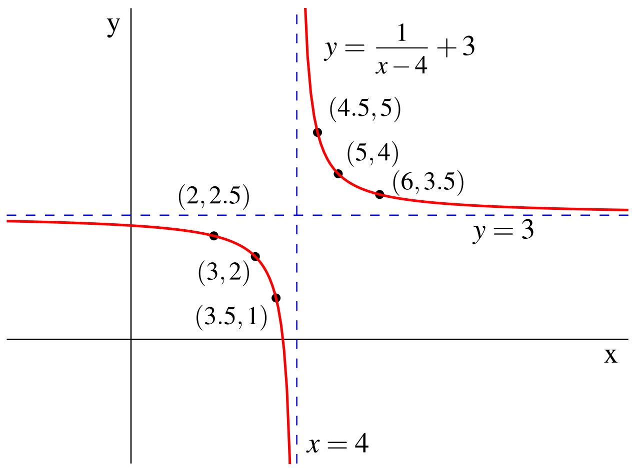


FIGURE 8.15. Graphing $y=\frac{1}{x-4}+3$

are translated to $x=h$ and $y=k$ where they are shown as dashed lines. Figure 8.16 on the next page illustrates this. Only one branch of the hyperbola is show in order to keep the picture simple.

In the shifted coordinate system (whose dashed axes are labeled x′ and y′), the coordinates of point P are $(1,1)$. In the non-shifted system with axes x and y, the coordinates of P are $(4,3)$.

Example 8.13

Graph the function $y=\dfrac{1}{x+3}+2$.

Solution

The horizontal shift is *left*, because $h<0$. In dashed coordinate system the graph of $\dfrac{1}{x}$ goes through points $P'(-1,-1)$ and $Q'(1,1)$. Since the dashed

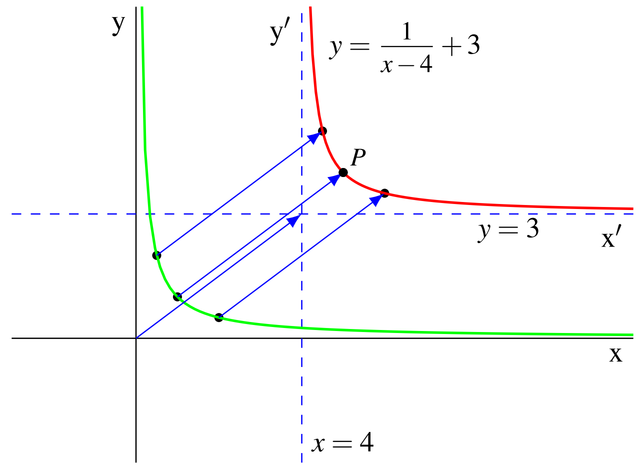


FIGURE 8.16. Graphing $y = \frac{1}{x-4} + 3$ by shifting coordinate system

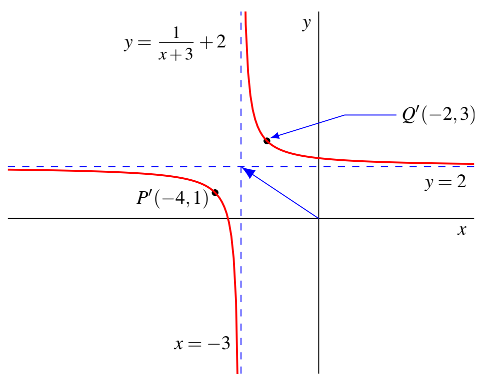


FIGURE 8.17. Graph $y = \frac{1}{x+3} + 2$. Shift 3 left, 2 up.

system is translated 3 left and 2 up, the location of P' in the xy-coordinate system is $(-4,1)$ and the location of Q' in the xy-coordinate system is $(-2,3)$. The graph is shown in Figure 8.17.

Example 8.14

Consider the function $y = \dfrac{3}{x+5} - 2$. State the domain, the range, and asymptotes, then graph the function.

Solution

Domain: all real numbers except -5. Range: all real numbers except -2. Asymptotes: $x = -5$ and $y = -2$. Figure 8.18 on the next page

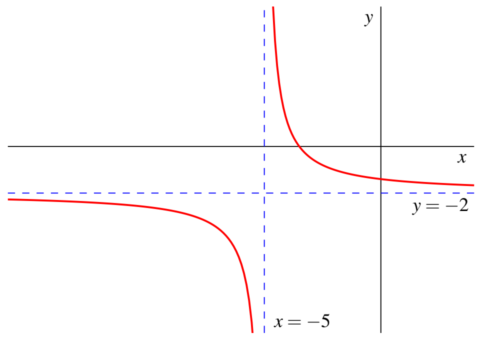


FIGURE 8.18. Graph $y = \frac{3}{x+5} - 2$.

Exercise 8.4

State the asymptotes, the domain, and the range.

1. $y = \dfrac{1}{x-6}$
2. $y = \dfrac{1}{x+3}$
3. $y = \dfrac{1}{x-7} + 1$
4. $y = \dfrac{1}{x-4} - 3$
5. $y = \dfrac{1}{2x+8}$
6. $y = \dfrac{3}{5x+20}$
7. $y = \dfrac{2}{3x-9} + 4$
8. $y = \dfrac{5}{7x+14} - 6$

Sketch the graph. Show asymptotes.

9. $f(x) = \dfrac{1}{x-2} - 3$
10. $f(x) = \dfrac{1}{x+1} + 2$

Chapter 9

Quadratic functions

9.1. The function $f(x) = x^2$

When $a \neq 0$, $ax^2 + bx + c$ is a quadratic expression, and $ax^2 + bx + c = 0$ is a quadratic equation. The function $f : f(x) = ax^2 + bx + c$ is new to you. We begin by considering the special case when $a = 1, b = 0$ and $c = 0$,

$$f(x) = x^2.$$

The function f is defined for all values of x. Since x^2 is never less than 0, the range of f is all nonnegative real numbers. As x takes larger positive values, $f(x)$ gets large without bound in the positive direction. And as x takes values farther to the left, $f(x)$ again becomes large without bound positive. If we let $y = f(x)$, the graph of f appears as in Figure 9.1.

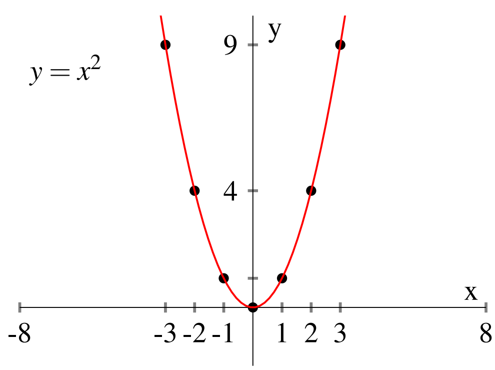


FIGURE 9.1. Graph $y = x^2$.

Methods to be certain of the shape of $y = x^2$ will be available to the student at a more advanced level. Plotting a lot of points will make the graph shown in Figure 9.1 plausible. We can be certain that no portion of $y = x^2$ is straight.

Let $P(x_1, y_1)$ be any fixed point on the graph of $y = x^2$. Let $Q(x, y)$ be any other (variable) point on the graph; that is, $x_1 \neq x$. Then

$$y_1 = x_1^2.$$

$$y = x^2.$$

$$\text{Slope}_{PQ} = \frac{x_1^2 - x^2}{x_1 - x}$$

$$= \frac{(x_1 - x)(x_1 + x)}{x_1 - x}$$

(9.1) $$\text{Slope}_{PQ} = x_1 + x.$$

But, as x takes various values, so does Slope_{PQ}. That is no one's idea of a constant slope.

Equation 9.1 does provide some idea of the shape of $y = x^2$. Notice that as x travels farther from x_1, $\text{Slope}_{PQ} = x_1 + x$ becomes steeper. Figure 9.2 shows example where $x_1 < x_2 < x_3$.

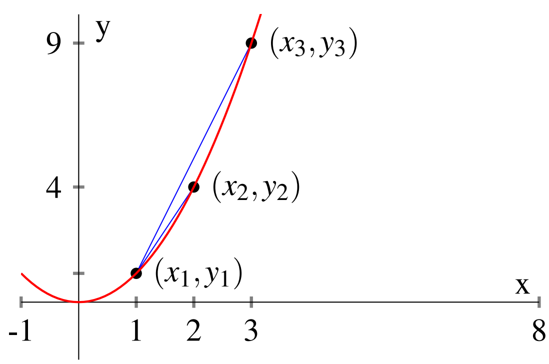


FIGURE 9.2. Graph $y = x^2$ is steeper as the distance of x from x_1 increases.

The graph of $y = x^2$ appears to have a line of symmetry. If in fact it does, that would be an interesting feature.

9.1.1. Symmetry of $f(x) = x^2$

Figure 9.3 on the following page captures our intuition of symmetry with respect to a line ℓ. Since PQ is perpendicular to ℓ and PR and QR are the same length, points P and Q are symmetric with respect to line ℓ.

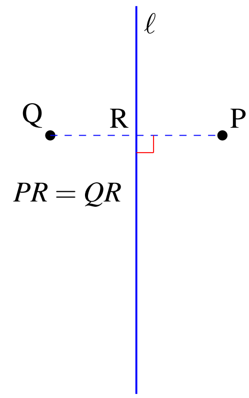


FIGURE 9.3. Points P and Q symmetric to line y.

Now consider $y = x^2$. Choose any point $P(x, y_1)$. Let Q be the point $Q(-x, y_2)$. Let R mark the point of intersection of PQ and the y-axis. See Figure 9.4.

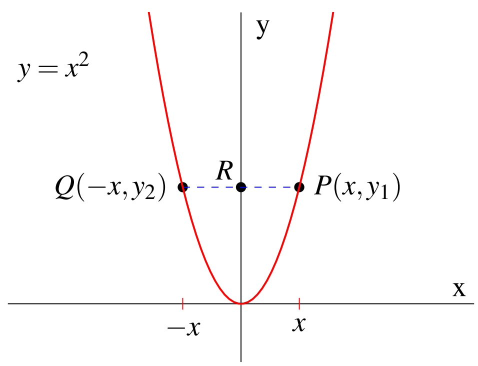


FIGURE 9.4. If PQ is perpendicular to the y-axis, $y = x^2$ is symmetric with respect to the y-axis..

$PR = QR$ because each is of length x. If PQ is perpendicular to the y-axis, then points P and Q are symmetric with respect to the y-axis. If PQ is parallel to the x-axis, then PQ is perpendicular to the y-axis. Now PQ is parallel to the x-axis, if $y_1 = y_2$. Here is how we know $y_1 = y_2$.

$$y_1 = x^2.$$

$$y_2 = (-x)^2 = x^2.$$

So,

$$y_1 = y_2.$$

Since for every point P on $y = x^2$ there is a point Q symmetric to P with respect to the y-axis, the curve $y = x^2$ is symmetric with respect to the y-axis.

We have proved part of the following theorem.

Theorem 9.1
Suppose $y = f(x)$ is defined on domain D. The function f is symmetric with respect to the y-axis if and only if $f(x) = f(-x)$ for every $x \in D$. ■

The curve $y = x^2$ is called a parabola. Every parabola has a line of symmetry that is sometimes called an "axis of symmetry". The point on the parabola at which the minimum (or maximum) value of the function occurs is called the "vertex". It is a fact that the vertex is on the line of symmetry of the parabola.

Exercise 9.1

Answer the following.

1. Show that no portion of the graph of $y = x^2$ is a straight line. See Section 8.2.2 on page 181.

2. Show that $y = x^4$ is symmetric with respect to the y-axis.

3. What point on $y = x^2$ is symmetric to $(3,9)$ with respect to the y-axis?

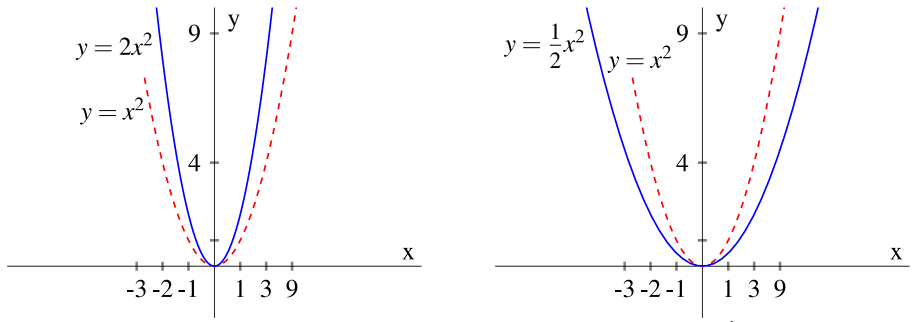
4. For $f(x) = x^2$, show that if the argument is tripled, the value of the function is increased by a factor of 9.

5. For $f(x) = x^2$, show that if the argument is halved, the value of the function is quartered.

6. Find the points at which the parabola x^2 and the line $y = x$ intersect.

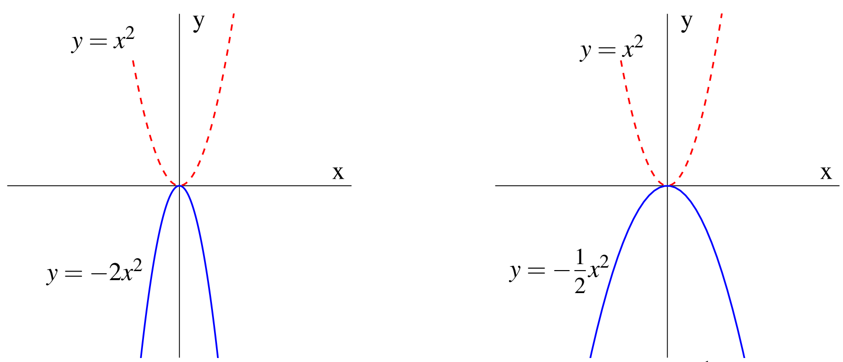
7. For what values of x does the graph of $y = x^2$ lie below the line $y = x$?

8. For what values of x does the graph of $y = x^2$ lie below the line $y = 5x$?



(a) $y = 2x^2$. (b) $y = \frac{1}{2}x^2$

FIGURE 9.5. Comparison to $y = x^2$ (dashed)



(a) $y = -2x^2$. (b) $y = -\frac{1}{2}x^2$

FIGURE 9.6. Comparison to $y = x^2$ (dashed)

9.2. The function $f(x) = ax^2$

When $a \neq 0$, $y = ax^2$ says that y is directly proportional to the square of x. The constant of proportionality is a. A common example is the area of a square which is directly proportional to the square of its side.

It is not hard to imagine the effect of the constant a. For $a > 1$, $ax^2 > x^2$. For $0 < a < 1$, $ax^2 < x^2$. If $a > 0$, the parabola is "concave up". If $a < 0$, the parabola is "concave down". Figures 9.5 to 9.6 on this page illustrate these cases.

A parabola such as $y = -2x^2$ that opens down is said to be "concave down". A parabola such as $y = 2x^2$ that opens up is called "concave up".

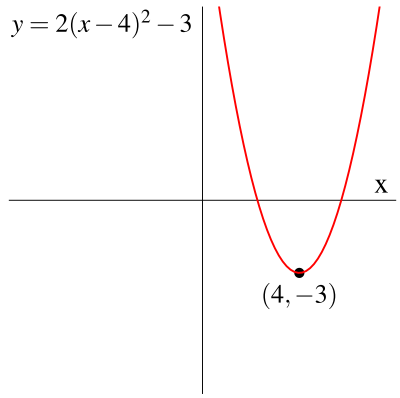


FIGURE 9.7. Graph $y = 2(x-4)^2 - 3$.

9.3. The function $f(x) = a(x-h)^2 + k$

After the discussion of $y = \dfrac{a}{x-k} + k$, you might suspect that $y = a(x-h)^2 + k$ is the parabola $y = ax^2$ translated h units horizontally and k units vertically. If so, you are correct. Figure 9.7 illustrates this.

9.4. The function $f(x) = ax^2 + bx + c$

9.4.1. Intercepts

When $a \neq 0$, $f(x) = ax^2 + bx + c$ is a parabola. Finding the x-intercepts just amounts to solving $ax^2 + bx + c = 0$ for x. If we desire to know the y-intercept, we compute $f(0)$.

9.4.2. Finding features of a quadratic function

We can rewrite $f(x) = ax^2 + bx + c$ in a form that reveals other characteristics. We use the familiar process of completing the square.

Let $f : f(x) = ax^2 + bx + c$ where $a \neq 0$. Then,

$$\begin{aligned} f(x) &= ax^2 + bx + c \\ &= a\left(x^2 + \frac{b}{a}x\right) + c \\ &= a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right) + c - a\left(\frac{b}{2a}\right)^2 \end{aligned}$$

$$= a\left(x+\frac{b}{2a}\right)^2 + c - a\left(\frac{b}{2a}\right)^2. \tag{9.2}$$

Since the real numbers are closed under the operations present in Equation 9.2, we may replace $\frac{b}{2a}$ by a single constant. We choose $-h$. We replace $c-a\left(\frac{b}{2a}\right)^2$ by k. The result is Equation 9.3

$$f(x) = a(x-h)^2 + k. \tag{9.3}$$

It is immediately clear that $f(x) = ax^2+bx+c$ is the parabola $y = ax^2$ translated h units horizontally and k units vertically. But there is more to observe. The Since $(x-h)^2$ is never less than 0, the minimum or maximum values for $f(x)$ occur when $x = h$. The minimum or maximum value must be k. If $a > 0$, then the graph is concave up and k is the minimum value of f. When $a < 0$, f is concave down making k the maximum value of f. Moreover, the line $x = h$ is the line of symmetry for the parabola. The location of the vertex in the coordinate plane is (h,k).

The x-intercepts of $f(x) = a(x-h)^2+k$ occur when $f(x) = 0$. We solve $a(x-h)^2+k = 0$ for x.

$$a(x-h)^2 + k = 0$$

$$(x-h)^2 = \frac{-k}{a}$$

$$x-h = \pm\sqrt{\frac{-k}{a}}$$

$$x = h \pm \sqrt{\frac{-k}{a}}. \tag{9.4}$$

Note that Equation 9.4 has solutions in the real numbers only when $\frac{-k}{a} \geq 0$. If you happen to remember Equation 9.4, you will be able to compute the x-intercepts quickly. If you forget Equation 9.4, then you will solve $ax^2+bx+c = 0$ by one of the methods you have learned.

Example 9.1

Given $f : f(x) = x^2+5x+6$. Graph f and find

(1) domain of f
(2) orientation of f (concave up or concave down)
(3) location of the vertex
(4) maximum or minimum value of $f(x)$

(5) range of f
(6) line of symmetry of f
(7) x-intercepts

Solution

Rewrite $f(x) = x^2 + 5x + 6$ in the form $f(x) = a(x-h)^2 + k$.

$$f(x) = x^2 + 5x + 6$$

$$= \left(x + \frac{5}{2}\right)^2 + 6 - \frac{25}{4}$$

$$= \left(x + \frac{5}{2}\right)^2 - \frac{1}{4} \tag{9.5}$$

By inspection of Equation 9.5,

(1) Domain of f is all real numbers.
(2) f is concave up. ($\because a > 0$)
(3) The vertex is at $\left(-\frac{5}{2}, -\frac{1}{4}\right)$.
(4) The minimum value attained by $f(x)$ is $-\frac{1}{4}$, $\because$ f is concave up.
(5) Range of f is all real numbers no less than $-\frac{1}{4}$.
(6) The line of symmetry is $x = -\frac{5}{2}$.

Solve for x when $f(x) = 0$ to get x-intercepts. Suppose we remember Equation 9.4 on the facing page, Then

$$x = -\frac{5}{2} \pm \sqrt{\frac{1}{4}}$$

$$= -\frac{5}{2} \pm \frac{1}{2}$$

$$\therefore x = -3 \text{ or } x = -2.$$

(7) The x-intercepts are -3 and -2

The graph of f is in Figure 9.8 on the next page where $y = f(x)$.

Remark 9.1

No law says you must use Equation 9.4 on the facing page to find the x-intercepts. Use whatever method easiest and least error prone for the particular equation you wish to solve. In Example 9.1 on the preceding page, simply factoring $x^2 + 5x + 6$ might have been best.

Remark 9.2

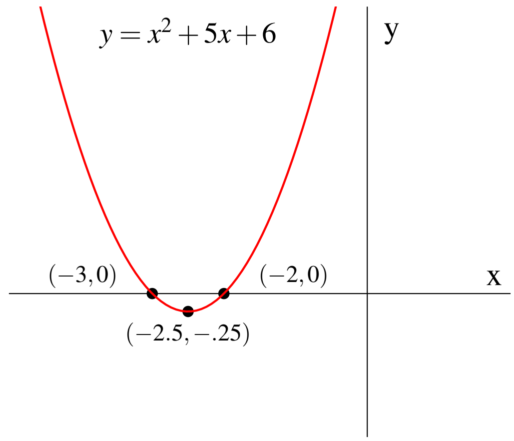


FIGURE 9.8. Graph $y = x^2 + 5x + 6$.

The wording of the question "Graph f and find . . . " implies a sequential nature to the work. That is unfortunate, because to arrive at the solution one fills in the details of the graph while one answers the several parts of the question. The non-sequential nature of the solution is hard to communicate in a book without producing copious annotated figures that make the process appear harder than it is.

Example 9.2

Given $f : f(x) = 2x^2 - 5x - 3$. Graph f and find

(1) domain of f
(2) orientation of f (concave up or concave down)
(3) location of the vertex
(4) maximum or minimum value of $f(x)$
(5) range of f
(6) line of symmetry of f
(7) x-intercepts

Solution

Rewrite $f(x) = 2x^2 - 5x - 3$ in the form $f(x) = a(x - h)^2 + k$.

$$\begin{aligned} f(x) &= 2x^2 - 5x - 3 \\ &= 2\left(x^2 - \frac{5}{2}x\right) - 3 \\ &= 2\left(x^2 - \frac{5}{2}x + \frac{25}{16}\right) - 3 - 2\left(\frac{25}{16}\right) \end{aligned}$$

(9.6) $$= 2\left(x - \frac{5}{4}\right)^2 - \frac{49}{8}$$

By inspection of Equation 9.6,

(1) Domain of f is all real numbers.
(2) f is concave up. $(\because a > 0)$
(3) The vertex is at $\left(\frac{5}{4}, -\frac{49}{8}\right)$.
(4) The minimum value attained by $f(x)$ is $-\frac{49}{8}$, $\because$ f is concave up.
(5) Range of f is all real numbers no less than $-\frac{49}{8}$.
(6) The line of symmetry is $x = \frac{5}{4}$.

Solve for x when $f(x) = 0$ to get x-intercepts. Recalling Equation 9.4 on page 206,

$$x = \frac{5}{4} \pm \sqrt{\frac{\frac{49}{8}}{2}}$$

$$= \frac{5}{4} \pm \sqrt{\frac{49}{16}}$$

$$= \frac{5}{4} \pm \frac{7}{4}$$

$$\therefore x = -\frac{1}{2} \text{ or } x = 3.$$

(6) The x-intercepts are $-\frac{1}{2}$ and 3.

The graph of f is in Figure 9.9 on the following page where $y = f(x)$.

Example 9.3

Given $f : f(x) = -2x^2 - 3x + 2$. Graph f and find

(1) domain of f
(2) orientation of f (concave up or concave down)
(3) location of the vertex
(4) maximum or minimum value of $f(x)$
(5) range of f
(6) line of symmetry of f
(7) x-intercepts

Solution

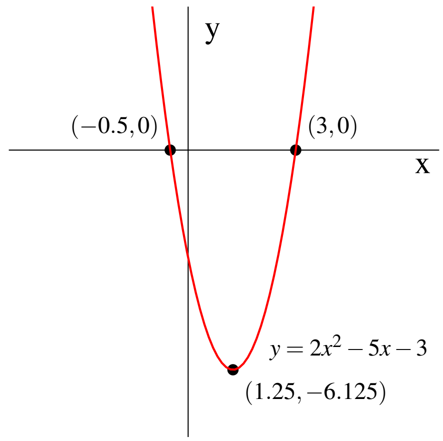


FIGURE 9.9. Graph $2x^2 - 5x - 3$.

Rewrite $f(x) = -2x^2 - 3x + 2$ in the form $f(x) = a(x-h)^2 + k$.

$$f(x) = -2x^2 - 3x + 2$$

$$= -2\left(x^2 + \frac{3}{2}x\right) + 2$$

$$= -2\left(x^2 + \frac{3}{2}x + \frac{9}{16}\right) + 2 + 2\left(\frac{9}{16}\right) \tag{9.7}$$

$$= -2\left(x + \frac{3}{4}\right)^2 + \frac{25}{8} \tag{9.8}$$

A favorite mistake is forgetting that the term which completes the square contributes a times that term to the equation. Nearly as popular a mistake is neglecting the sign of a. The mistakes would have occurred at Equation 9.7.

By inspection of Equation 9.8,

(1) Domain of f is all real numbers.
(2) f is concave down. $(\because a < 0)$
(3) The vertex is at $\left(\frac{3}{4}, \frac{25}{8}\right)$.
(4) The maximum value attained by $f(x)$ is $\frac{25}{8}$, $\because$ f is concave down.
(5) Range of f is all real numbers no greater than $\frac{25}{8}$.

Use Equation 9.4 on page 206 to find the x-intercepts.

$$x = -\frac{3}{4} \pm \sqrt{\frac{\frac{25}{8}}{2}}$$

$$= -\frac{3}{4} \pm \sqrt{\frac{25}{16}}$$

$$= -\frac{3}{4} \pm \frac{5}{4}$$

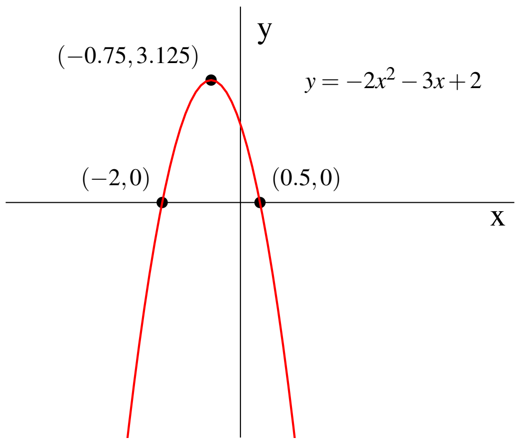


FIGURE 9.10. Graph of $y = -2x^2 - 3x + 2$.

$$\therefore x = \frac{1}{2} \text{ or } x = -2.$$

(6) The graph of $y = -2x^2 - 3x + 2$ intersects the x-axis ate $x = -2$ and $x = 1/2$.

The graph of f is in Figure 9.10 where $y = f(x)$.

■

At Equation 9.7 on the preceding page, we *added* $2\left(\frac{9}{16}\right)$ outside the parenthesis. That is because a is negative. Completing the square by adding $\frac{9}{16}$ inside the parenthesis amounted to subtracting $2\left(\frac{9}{16}\right)$ from the equation. Adding $2\left(\frac{9}{16}\right)$ outside the parenthesis makes the net addition to the equation 0.

A favorite mistake is forgetting that the term which completes the square contributes a times that term to the equation. Nearly as popular a mistake is neglecting the sign of a. The mistakes would have occurred at Equation 9.7 on the facing page.

Example 9.4

Given $f : f(x) = x^2 + 5x + 7$. Graph f and find

(1) domain of f
(2) orientation of f (concave up or concave down)
(3) location of the vertex
(4) maximum or minimum value of $f(x)$
(5) range of f
(6) line of symmetry of f

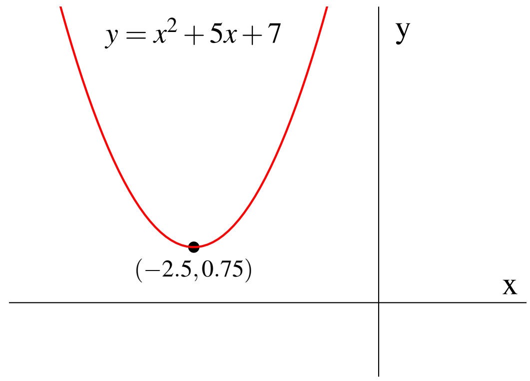


FIGURE 9.11. Graph $x^2 + 5x + 7$.

(7) x-intercepts

Solution

Rewrite $x^2 + 5x + 7$ in the form $f(x) = a(x-h)^2 + k$.

$$f(x) = x^2 + 5x + 7$$

$$= \left(x + \frac{5}{2}\right)^2 + 7 - \frac{25}{4}$$

$$= \left(x + \frac{5}{2}\right)^2 - \frac{3}{4} \tag{9.9}$$

By inspection of Equation 9.9,

(1) Domain of f is all real numbers.
(2) f is concave up. ($\because a > 0$)
(3) The vertex is at $\left(-\frac{5}{2}, \frac{3}{4}\right)$.
(4) The minimum value attained by $f(x)$ is $\frac{3}{4}$, $\because$ f is concave up.
(5) Range of f is all real numbers no less than $\frac{3}{4}$.
(6) The line of symmetry is $x = -\frac{5}{2}$.

Solve for x when $f(x) = 0$ to get x-intercepts. Using Equation 9.4 on page 206,

$$x = -\frac{5}{2} \pm \sqrt{-\frac{3}{4}}$$

Which is undefined.

(7) The x-intercepts do not exist.

The graph of f is in Figure 9.11 where $y = f(x)$.

Example 9.5

State the orientation of f when $f(x) = 17x^2 + 24x - 16$.

Solution

By inspection, the coefficient of x^2 is positive. Therefore, f is concave up.

Example 9.6

Find the x-intercepts of $f(x) = x^2 - 2x + \frac{1}{4}$.

Solution

There is no need of all the "*h*-*k*" stuff. All we need are the x-intercepts. Just solve $f(x) = 0$ for x by any convenient method.

$$x^2 - 2x + \frac{1}{4} = 0$$

$$4x^2 - 8x + 1 = 0.$$

Then, using quadratic formula

$$x = \frac{8 \pm \sqrt{64 - 4(4)(1)}}{2(4)}$$

$$x = \frac{2 \pm \sqrt{3}}{2}.$$

The x-intercepts are $x = \frac{2+\sqrt{3}}{2}$ and $x = \frac{2-\sqrt{3}}{2}$.

Remark 9.3

Suppose Example 9.6 had asked for a graph. The exact values $\frac{2 \pm \sqrt{3}}{2}$ are not too handy for graphing. It is hard to know where to place them. Rational approximations are useful. Using a calculator, $\frac{2-\sqrt{3}}{2} \approx 1.3$ and $\frac{2+\sqrt{3}}{2} \approx 1.9$. An approximation to the nearest 0.1 is adequate for our purpose.

Exercise 9.2

Rewrite each function in the form $f(x) = (x-h)^2 + k$.

1. $f(x) = x^2 + 12x + 30$
2. $f(x) = x^2 - 10x + 34$
3. $f(x) = x^2 - 14x + 56$
4. $f(x) = 3x^2 - 12x + 12$
5. $f(x) = -x^2 + 6x - 4$
6. $f(x) = -2x^2 - 4x - 1$
7. $f(x) = 3x^2 + 12x + 5$
8. $f(x) = 3x^2 - 8$
9. $f(x) = -3x^2 - 24x - 58$
10. $f(x) = x^2 + 12x + 35$

Is the orientation concave up of concave down.

11. $f(x) = -\frac{1}{4}x^2 + 3x - 6$
12. $f(x) = -x^2 + 10x - 31$
13. $f(x) = x^2 + 12x + 35$
14. $f(x) = 7x^2 - 10x - 25$

State the vertex and the orientation of each parabola.

15. $f(x) = x^2 - 6x - 1$
16. $f(x) = -\frac{1}{4}x^2 - 2x - 13$
17. $f(x) = x^2 + 16x + 62$
18. $f(x) = -4x^2 - 16x - 24$
19. $f(x) = -3x^2 - 30x - 76$
20. $f(x) = -5x^2 + 1$
21. $f(x) = -x^2 - 14x - 42$
22. $f(x) = 4x^2 - 32x + 61$
23. $f(x) = -x^2 + 14x - 56$
24. $f(x) = x^2 - 8x + 16$

Find the maximum (max) or minimum (min) value of the function. Be sure to say whether it is max or min.

25. $f(x) = x^2 + 8x + 18$
26. $f(x) = x^2 - 6x + 10$
27. $f(x) = x^2 + 2x + 5$
28. $f(x) = x^2 + 2x$
29. $f(x) = 3x^2 - 24x + 49$
30. $f(x) = -2x^2 - 12x - 25$
31. $f(x) = -x^2 + 8x - 8$
32. $f(x) = \frac{1}{3}x^2 + 6x + 18$
33. $f(x) = 2x^2 - 12x + 17$
34. $f(x) = -2x^2 + 12x - 22$

State the range of each function.

35. $f(x) = -\frac{1}{3}x^2 - 2x$
36. $f(x) = x^2 - 16x + 60$
37. $f(x) = -x^2 + 2x - 4$
38. $f(x) = -x^2 - 12x - 33$
39. $f(x) = x^2 - 6x + 3$
40. $f(x) = -2x^2 - 8x - 15$
41. $f(x) = x^2 - 2x - 9$
42. $f(x) = 2x^2 - 24x + 65$
43. $f(x) = \frac{5}{6}x^2 - \frac{10}{3}x + \frac{28}{3}$
44. $f(x) = x^2 + 20x + 94$

State the vertex and x-intercepts. Sketch the graph.

45. $f(x) = \frac{1}{4}x^2 - 1$
46. $f(x) = -2x^2 + 8x - 9$
47. $f(x) = -x^2 - 6x - 5$
48. $f(x) = 2x^2 + 12x + 10$
49. $f(x) = x^2 + 4x + 3$
50. $f(x) = -2x^2 + 24x - 73$
51. $f(x) = -\frac{1}{2}x^2 - x - \frac{3}{2}$
52. $f(x) = x^2 + 6x + 12$
53. $f(x) = 2x^2 + 20x + 48$
54. $f(x) = -x^2 + 4x - 5$

9.4.3. A simple way to find the vertex

Let $y = f(x) = ax^2 + bx + c, a \neq 0$. Providing the parabola intersects the x-axis, there is a very practical way to find the vertex of a parabola. Because of the symmetry of a parabola, the x-coordinate at the vertex must be midway between the x-intercepts. If the horizontal intercepts are x_1 and x_2, then the first coordinate, x, at the vertex must be

$$\bar{x} = \frac{x_1 + x_2}{2}.$$

The second coordinate is found by evaluating $f(x)$ at $x = \bar{x}$.

Example 9.7

Find the vertex of $y = x^2 - 9x + 14$.

Solution

Since $y = x^2 - 16x + 28 = (x-2)(x-14)$, the x-intercepts are $x = 2$ and $x = 14$. Compute $\bar{x} = \dfrac{2+14}{2} = 8$. Then $f(8) = -36$. So the vertex is located at $(8, -36)$.

Example 9.8

Find the vertex of $f(x) = x^2 - x - \dfrac{1}{2}$.

Solution

Of course, first we try to factor $x^2 - x - \dfrac{1}{2}$. Then we give up and use the quadratic formula. We find the roots of $x^2 - x - \dfrac{1}{2} = 0$ are $x = \dfrac{1+\sqrt{3}}{2}$ and $x = \dfrac{1-\sqrt{3}}{2}$. So

$$\bar{x} = \frac{1}{2}\left(\frac{1+\sqrt{3}}{2} + \frac{1-\sqrt{3}}{2}\right) = \frac{1}{2}.$$

Then $f\left(\dfrac{1}{2}\right) = -\dfrac{3}{4}$. The vertex is at $\left(\dfrac{1}{2}, -\dfrac{3}{4}\right)$.

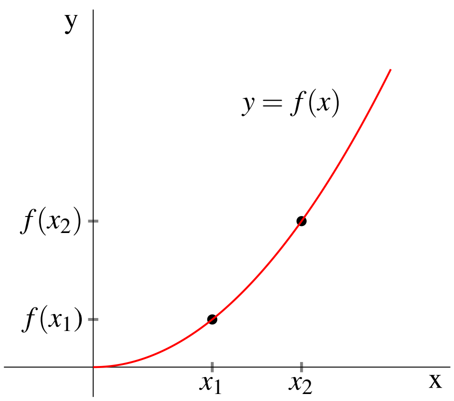
Example 9.9

Find the vertex of $f(x) = x^2 + 8x + 20$.

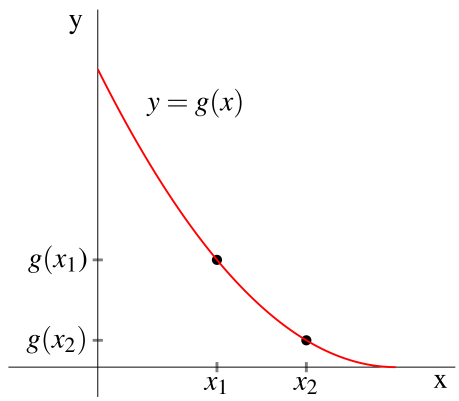
Solution

Before trying the "simple way", mentally compute the discriminate $b^2 - 4ac$. Since $64 < 80$, the function has no x-intercepts. The simple method would lead to a dead end.

Rewrite $f(x) = x^2 + 8x + 20$ as $f(x) = (x+4)^2 + 4$. By inspection, the vertex is at $(-4, 4)$.



(a) $y = f(x)$, f increasing



(b) $y = g(x)$, g decreasing

FIGURE 9.12. One function is decreasing, the other increasing.

Exercise 9.3

Using any convenient method, find the x-intercepts. If the exact answer is not an integer, then also provide the decimal approximation you would use for graphing. You are welcome to use a calculator for the approximations. If the are no x-intercepts, say so.

1. $f(x) = x^2 - 5x - 24$
2. $f(x) = x^2 - 2$
3. $f(x) = x^2 - 6x + 7$
4. $f(x) = x^2 - 2x - 4$
5. $f(x) = x^2 + 4x + 4$
6. $f(x) = x^2 + 4x + 5$
7. $f(x) = x^2 - 4x - 2$
8. $f(x) = 6x^2 + 7x + 2$
9. $f(x) = 6x^2 + 32x + 32$
10. $f(x) = x^2 + 6x + 10$

9.5. Increasing and decreasing functions

Consider two functions of x, one called f and the other called g. Suppose their graphs are as shown in Fiqure 9.12.

The words we use to express the obvious distinction between f and g are "increasing" and decreasing". We say that f is an increasing function and g is a decreasing function. When we describe f as an increasing function,

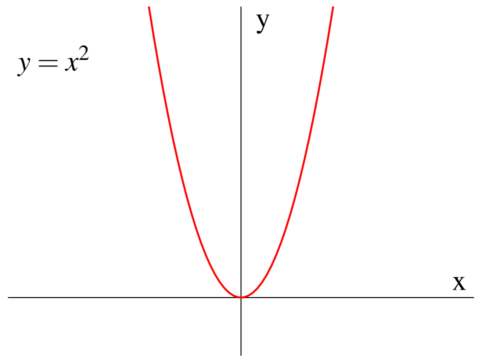


FIGURE 9.13. Graph $y = x^2$.

we mean that as the values of x become large, the corresponding values $f(x)$ become large, too. The function g appears decreasing, because as x becomes large, $g(x)$ becomes small.

Definition 9.1
A function f is called **increasing on it's domain D** if

$$x_2 > x_1 \quad \text{implies} \quad f(x_2) > f(x_1) \quad \text{for } x_1 \text{ and } x_2 \text{ in } D.$$

A function f is called **decreasing on it's domain D** if

$$x_2 > x_1 \quad \text{implies} \quad f(x_2) < f(x_1) \quad \text{for } x_1 \text{ and } x_2 \text{ in } D. \qquad \blacksquare$$

The function $f(x) = x^2$ domain all real numbers appears to be both increasing and decreasing on its domain. Figure 9.13.

We can refine Definition 9.1 to apply to a portion of a function's domain.

Definition 9.2
A function f is called **increasing on an interval I** if

$$x_2 > x_1 \quad \text{implies} \quad f(x_2) > f(x_1) \quad \text{for } x_1 \text{ and } x_2 \text{ in } I.$$

A function f is called **decreasing on an interval I** if

$$x_2 > x_1 \quad \text{implies} \quad f(x_2) < f(x_1) \quad \text{for } x_1 \text{ and } x_2 \text{ in } I. \qquad \blacksquare$$

Using Definition 9.2, we can say that $f(x) = x^2$ is decreasing for all $x < 0$ and increasing for all $x > 0$. In fact, we can prove this is so.

To prove that $f(x) = x^2$ is decreasing on interval $x < 0$, suppose that $x_1 < x_2 < 0$. Consider the difference $f(x_2) - f(x_1)$.

$$\begin{aligned} f(x_2) - f(x_1) &= x_2^2 - x_1^2 \\ &= (x_2 + x_1)(x_2 - x_1). \end{aligned}$$

but (x_2+x_1) is negative and (x_2-x_1) is positive. So,

$$f(x_2)-f(x_1)<0$$

This means that $f(x_2)<f(x_1)$. Therefore, by definition, $f(x)=x^2$ is a decreasing function of x on interval $x<0$.

The proof that $f(x)$ is increasing on $x>0$ is left as an exercise.

Remark 9.4
Notice where we used the supposition that x_1 and x_2 are in interval $x<0$.

It is a little harder to show that $f(x)=x^2+5x+6$ is decreasing to the left of $x=-3$. We do so in the next example.

Example 9.10

Show that $f:f(x)=x^2+6x+8$ is decreasing for $x<-3$.

Solution

Let $f(x)=x^2+6x+8$. Suppose that $x_1<x_2<-3$. We plan to show the difference $f(x_2)-f(x_1)$ is negative, then conclude that $f(x_2)<f(x_1)$.

$$\begin{aligned}
f(x_2)-f(x_1) &= x_2^2+6x_2+8-(x_1^2+6x_1+8)\\
&= x_2^2-x_1^2+6x_2-6x_1\\
&= (x_2-x_1)(x_2+x_1)+6(x_2-x_1)\\
&= (x_2-x_1)(x_2+x_1+6).
\end{aligned}$$

Note that $x_1<-3$ and $x_2<-3$. So that $x_2+x_1<-6$. This guarantees that (x_2+x_1+6) is negative. Of course, (x_2-x_1) is positive because we supposed $x_1<x_2$. Consequently,

$$f(x_2)-f(x_1)<0.$$

Therefore, $f(x_2)<f(x_1)$. This makes f a decreasing function for $x<-3$.

Exercise 9.4

1. Show that $f(x) = x^2$ is increasing on the interval $x > 0$.
2. Show that $f(x) = x^2 - 10x + 16$ is increasing on the interval $x > 5$.
3. Based on the *appearance* of the graph $y = f(x)$, state the intervals on which the function is increasing or decreasing. See Figure 9.14.

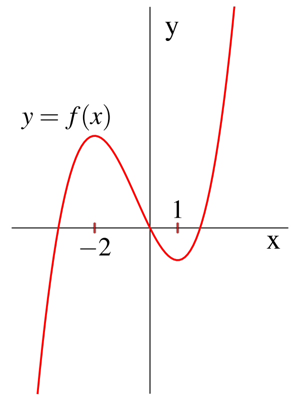


FIGURE 9.14. For question 3

4. Based on the *appearance* of the graph $y = f(x)$, state the intervals on which the function is increasing or decreasing. See Figure 9.15.

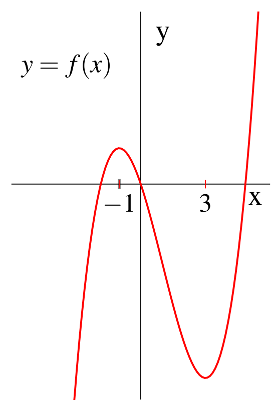


FIGURE 9.15. For question 4

5. Show that $y = mx^2 + b$, $m \neq 0$ is a constant, is nowhere straight.
6. Show that graph of $y = mx^2 + bx + c$, $m \neq 0$ is a constant, is nowhere straight.

Chapter 10

The idea of a function

Mathematicians explored the concept of function for decades of the 19th century. The definition that finally emerged captured the essential nature of a function. Several qualities of a function uncovered in those years, while not definitive, enhance our understanding of functions. *Beginning Algebra* emphasized these non-definitive qualities.

Here is the definition from *Beginning Algebra*.

> A **function** is a rule that shows how the value of one variable, called the **dependent variable**, is uniquely determined by the value of another variable, called the **independent variable**.

The time has come to stress the "uniquely determined" part of this definition. It is the defining feature of a function in the following definition.

Definition 10.1 (Function)
A **function** is a set f of ordered pairs such that if x is the first coordinate of an ordered pair in f, then there is exactly one y such that $(x,y) \in f$. ■

Sometimes Definition 10.1 is paraphrased by saying "a function is a collection of ordered pairs such that no first element appears twice".

For example, let set $A = \{(2,5),(-3,8),(7,13),(5,-12)\}$. Set A is a function. Well, A satisfies Definition 10.1, doesn't it?

On the other hand, let set $B = \{(1,3),(2,4),(3,5),(1,7),(11,17)\}$. Set B is not a function. It fails because 1 occurs in the two ordered pairs $(1,3)$ and $(1,7)$.

Suppose set $C = \{(-3,5),(-2,5),(-1,5),(0,5),(1,5),(2,5)\}$. Is C a function? Yes. Although the same second coordinate appears in every ordered pair, no first coordinate appears more than once.

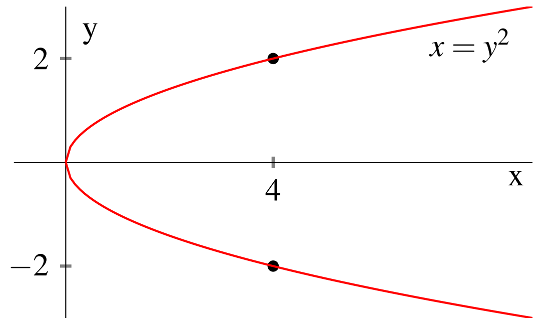


FIGURE 10.1. Graph of $x = y^2$. No function.

Until now, every equation you have met that related two variables has been a function. Let us consider several relations that are not functions.

You are familiar with the parabola that is concave up (opens up) or concave down (opens down). A simple equation for such a parabola is

$$y = x^2.$$

Now consider the equation of a parabola that opens to the right.

$$x = y^2. \tag{10.1}$$

Does Equation 10.1 also define y as a function of x?

The graph of Equation 10.1 is shown in Figure 10.1.

The graph of Equation 10.1 shows one value of x paired with two values of y. In other words, the ordered pairs $(4,2)$ and $(4,-2)$ belong to the same set of points. By Definition 10.1 on the preceding page, that set of points cannot be a function. Equation 10.1 does not define y as a function of x.

There is a visual test called the "vertical line" test. We just performed it. The vertical line test says that if any vertical line cuts a graph more than once, then the graph is not of a function. Imagine a vertical line through $x = 4$. It cuts $x = y^2$ twice, once at $(4,-2)$ and again at $(4,2)$.

Another classic example of a relation that fails to be a function is the equation

$$x^2 + y^2 = 4. \tag{10.2}$$

The graph of Equation 10.2 is the cute circle radius 2 that appears in Figure 10.2 on the next page. But, cute or otherwise, the circle fails miserably the vertical line test. The dashed vertical line shown cuts the circle twice.

The vertical line test is handy. And persuasive. But it is no proof.

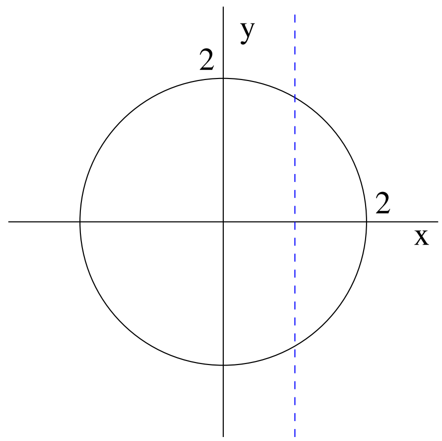


FIGURE 10.2. Graph of $x^2+y^2=4$. No function.

Example 10.1

Prove that the relation $x=y^2$ does not define y as a function of x.

Solution

We need only produce a single instance where two values of y are paired with one value of x. The pair $(4,-2)$ satisfies the equation, because $4 = (-2)^2$. But the pair $(4,2)$ also satisfies the equation, because $4=2^2$. Since one value of x is paired with two values of y, the equation $x=y^2$ does not define y as a function of x.

Example 10.2

Prove that $y=x^2$ defines y as a function of x.

Proof. Let $y=x^2$. Suppose that x_1 is paired with y_1 and that x_2 is paired with y_2. That is, $y_1=x_1^2$ and $y_2=x_2^2$. We plan to show that if the first coordinates of (x_1,y_1) and (x_2,y_2) are the same the second coordinates must also be identical. So, suppose x_1=x_2. Then

$$\begin{aligned} x_1 = x_2 &\implies x_1^2 = x_2^2 \\ &\implies y_1 = y_2. \end{aligned}$$

■

The proof in Example 10.2 seemed so slick, that we almost wonder if we could "prove" $x=y^2$ defines y as a function of x. If we can, then something is wrong.

"Proof". Let $x=y^2$. Suppose that x_1 is paired with y_1 and that x_2 is paired with y_2. That is, $x_1=y_1^2$ and $x_2=y_2^2$. We plan to show that if the first coordinates of (x_1,y_1) and (x_2,y_2) are the same the second coordinates must also be identical. So, suppose x_1=x_2. Then

$$x_1 = x_2 \implies y_1^2 = y_2^2.$$

To finish the "proof", we must now say

$$\implies y_1 = y_2.$$

But we cannot. Why not?

We cannot pass to the next line, because $y_1^2 = y_2^2$ does not necessarily imply that $y_1 = y_2$. The best we could conclude from $y_1^2 = y_2^2$ is that either $y_1 = y_2$ *or* $y_1 = -y_2$. And that is not good enough.

Exercise 10.1

Answer the following.

1. Prove that $y = 2x + 3$ defines y as a function of x.
2. Restrict the domain of $y = \frac{1}{x}$ to $x > 0$. Show that $y = \frac{1}{x}$ on this restricted domain defines y as a function of x.
3. A quadratic function has the form $ax^2 + bx + c$, $a \neq 0$. Show that if f and g are quadratic functions and $h(x) = f(x) + g(x)$ the h is a quadratic function.
4. Suppose $f(x) = mx + k$. Under what condition does $f(a+b) = f(a) + f(b)$ for all numbers a and b?
5. Let x represent any real number. Is $y = 9$ a function?
6. Provide an example showing that $y_1^2 = y_2^2$ does not necessarily imply that $y_1 = y_2$.

Appendices

Appendix A

Sieve of Eratosthenes

Sieve of Eratosthenes (10×10)

Step 1: Numbers from 2 . . . 100

	2	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
11	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
21	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**
31	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**
41	**42**	**43**	**44**	**45**	**46**	**47**	**48**	**49**	**50**
51	**52**	**53**	**54**	**55**	**56**	**57**	**58**	**59**	**60**
61	**62**	**63**	**64**	**65**	**66**	**67**	**68**	**69**	**70**
71	**72**	**73**	**74**	**75**	**76**	**77**	**78**	**79**	**80**
81	**82**	**83**	**84**	**85**	**86**	**87**	**88**	**89**	**90**
91	**92**	**93**	**94**	**95**	**96**	**97**	**98**	**99**	**100**

Step 2: Eliminated multiples of **2**

	2	**3**		**5**		**7**		**9**	
11		**13**		**15**		**17**		**19**	
21		**23**		**25**		**27**		**29**	
31		**33**		**35**		**37**		**39**	
41		**43**		**45**		**47**		**49**	
51		**53**		**55**		**57**		**59**	
61		**63**		**65**		**67**		**69**	
71		**73**		**75**		**77**		**79**	
81		**83**		**85**		**87**		**89**	
91		**93**		**95**		**97**		**99**	

Primes:
2

Step 3: Eliminated multiples of **3**

	2	**3**		**5**		**7**			
11		**13**				**17**		**19**	
		23		**25**				**29**	
31				**35**		**37**			
41		**43**				**47**		**49**	
		53		**55**				**59**	
61				**65**		**67**			
71		**73**				**77**		**79**	
		83		**85**				**89**	
91				**95**		**97**			

Primes:
2, 3

Step 4: Eliminated multiples of **5**

	2	**3**		**5**		**7**			
11		**13**				**17**		**19**	
		23						**29**	
31						**37**			
41		**43**				**47**		**49**	
		53						**59**	
61						**67**			
71		**73**				**77**		**79**	
		83						**89**	
91						**97**			

Primes:
2, 3, 5

Step 5: Eliminated multiples of **7**

	2	**3**		**5**		**7**			
11		**13**				**17**		**19**	
		23						**29**	
31						**37**			
41		**43**				**47**			
		53						**59**	
61						**67**			
71		**73**						**79**	
		83						**89**	
						97			

Primes:
2, 3, 5, 7

Step 25: Remaining are prime.

	2	**3**		**5**		**7**			
11		**13**				**17**		**19**	
		23						**29**	
31						**37**			
41		**43**				**47**			
		53						**59**	
61						**67**			
71		**73**						**79**	
		83						**89**	
						97			

Primes:
2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97

Appendix B

Answers to Exercises

Answers to Exercise 1.1

(**1**)
(**a**) $1, 2, 3, 6$
(**b**) $1, 3, 7, 21$
(**c**) $1, 2, 3, 4, 6, 12$
(**d**) $1, 3, 5, 9, 15, 45$
(**e**) $1, 2, 3, 4, 6, 9, 12, 18, 36$
(**f**) $1, 7$

(**2**) (**a**) $5, 10, 15, 20, 25, 30$
(**b**) $6, 12, 18, 24, 30, 36$
(**c**) $1, 2, 3, 4, 5, 6$
(**d**) $10, 20, 30, 40, 50, 60$
(**e**) $7, 14, 21, 28, 35, 42$
(**f**) $101, 202, 303, 404, 505, 606$

(**3**) Yes.
(**4**) No.
(**5**) 996
(**6**) $23 \div 7 = 3$ with remainder $2 \neq 0$.
(**7**) 4
(**8**) 15
(**9**) 9
(**10**) 4
(**11**) $2n - 1, n = 1, 2, 3, \cdots$ produces all and only odd numbers.

Answers to Exercise 1.2

(**1**)
(**a**) 2^5
(**b**) $2^3\,5^2$
(**c**) $7 \cdot 11 \cdot 13^2\,19$
(**d**) $2 \cdot 3 \cdot 7^4$
(**e**) $2^2 3^2 5^3 11^2$

(**2**)
(**a**) $2 \cdot 3 \cdot 5 \cdot 11$
(**b**) $2^2\,7$
(**c**) $2^3\,11$
(**d**) $5 \cdot 7$
(**e**) $5^2\,13$
(**f**) $2 \cdot 3^2\,5$
(**g**) $3 \cdot 5^3\,11$
(**h**) $2^6 3^3$
(**i**) $2 \cdot 3^3\,7^2$
(**j**) $2^2 \cdot 3 \cdot 11$

(**3**)	(**e**) 16	(**j**) 125
(**a**) 40	(**f**) 100	(**k**) 600
(**b**) 27	(**g**) 100	(**l**) 588
(**c**) 54	(**h**) 1000	
(**d**) 36	(**i**) 10000	

Answers to Exercise 1.3

(**1**)	(**e**) 21	(**j**) 16
(**a**) 16	(**f**) 13	(**k**) 6
(**b**) 24	(**g**) 14	(**l**) 24
(**c**) 14	(**h**) 6	
(**d**) 19	(**i**) 14	

(**2**)	(**e**) 96	(**j**) 38
(**a**) 42	(**f**) 12	(**k**) 11
(**b**) 35	(**g**) 21	(**l**) 66
(**c**) 32	(**h**) 2	
(**d**) 1	(**i**) 58	

(**3**)	(**e**) 14	(**j**) 23
(**a**) 1	(**f**) 17	(**k**) 17
(**b**) 6	(**g**) 16	(**l**) 16
(**c**) 18	(**h**) 11	
(**d**) 12	(**i**) 8	

(**4**) $(4,28),(12,20)$. (**5**) $(90,234),(126,198)$

Answers to Exercise 1.4

(**1**)	(**e**) 120	(**j**) 90
(**a**) 180	(**f**) 42	(**k**) 48
(**b**) 144	(**g**) 120	(**l**) 144
(**c**) 72	(**h**) 220	
(**d**) 84	(**i**) 84	

(**2**)	(**c**) 144	(**f**) 240
(**a**) 300	(**d**) 132	
(**b**) 630	(**e**) 198	

(**3**) 56 (**4**) $(2,90),(10,18)$ (**5**) 42 days

Answers to Exercise 1.5

(**1**) 12
(**2**) 8
(**3**) 12
(**4**) 20
(**5**) $1,2,4,8$
(**6**) $1,2,7,14,49,98$
(**7**) $1,2,3,6,9,18,27,54$
(**8**) $1,3,5,9,15,25,27,45,75,135,225,675$

Answers to Exercise 2.1

(**1**) $y-8=\frac{2}{3}(x-3)$ or $y-14=\frac{2}{3}(x-12)$

(**2**) $y+1=-\frac{3}{4}(x-4)$ or $y+7=-\frac{3}{4}(x-12)$

(**3**) $y+2=-\frac{1}{7}(x+7)$ or $y+5=-\frac{1}{7}(x-14)$

(**4**) $y-15=-3(x+3)$ or $y+6=-3(x-4)$

(**5**) $y+13=4(x+2)$ or $y-11=4(x-4)$

(**6**) $y-1=-\frac{7}{3}x$ or $y+27=-\frac{7}{3}(x-12)$

(**7**) $y=\frac{2}{15}x+\frac{14}{3}$

(**8**) $y=\frac{1}{4}x-3$

(**9**) $y=\frac{-8}{7}x+\frac{151}{7}$

(**10**) $y=-\frac{1}{2}x+7$

(**11**) $y=\frac{3}{4}x+2$

(**12**) $y=-\frac{2}{3}x-8$

(**13**) $6x+y=7$

(**14**) $2x+1y=1$

(**15**) $7x-2y=8$

(**16**) $x+5y=-13$

(**17**) $7x-8y=5$

(**18**) $7x+4y=-8$

(**19**) $4x-y=-20$

(**20**) $7x+y=4$

(**21**) $7x+8y=-19$

(**22**) $5x+4y=-17$

(**23**) $3x-4y=-16$

(**24**) $2x+3y=10$

(**25**) slope $=-\frac{2}{5}$, x-int $=5$, y-int $=2$.

(**26**) slope $=-\frac{5}{3}$, x-int $=-3$, y-int $=-5$.

(**27**) slope $=-\frac{2}{5}$, x-int $=-5$, y-int $=-2$.

(**28**) slope $=-\frac{1}{7}$, x-int $=7$, y-int $=1$.

(**29**) slope $= -\frac{4}{5}$, x-int $= 5$, y-int $= 4$.

(**30**) slope $= 1$, x-int $= 1$, y-int $= -1$.

(**31**) slope $= \frac{3}{2}$, x-int $= 2$, y-int $= -3$.

(**32**) slope $= \frac{2}{3}$, x-int $= 3$, y-int $= -2$.

(**33**) slope $= -\frac{3}{2}$, x-int $= \frac{2}{3}$, y-int $= 1$.

(**34**) slope $= \frac{5}{2}$, x-int $= \frac{2}{5}$, y-int $= -1$.

(**35**) slope $= -\frac{1}{3}$, x-int $= -1$, y-int $= \frac{-1}{3}$.

(**36**) slope $= -\frac{3}{10}$, x-int $= \frac{-2}{3}$, y-int $= \frac{-1}{5}$.

(**37**) slope $= -\frac{7}{2}$, x-int $= -2$, y-int $= -7$.

(**38**) slope $= 5$, x-int $= -1$, y-int $= 5$.

(**39**) slope $= -\frac{12}{13}$, x-int $= 13$, y-int $= 12$.

(**40**) slope $= -2$, x-int $= -2$, y-int $= -4$.

Answers to Exercise 2.2

(**1**) $x+4y=-4$

(**2**) $6x-y=5$

(**3**) $x-y=2$

(**4**) $3x+y=-3$

(**5**) $x-y=0$

(**6**) $7x-5y=25$

(**7**) $8x+y=-5$

(**8**) $x-y=-1$

(**9**) $3x-y=3$

(**10**) $x=-5$

(**11**) $8x+3y=-9$

(**12**) $3x+5y=20$

(**13**) $x+2y=-4$

(**14**) $3x-5y=-16$

(**15**) $8x+5y=9$

(**16**) $3x+y=-1$

(**17**) $10x-9y=-5$

(**18**) $7x-3y=-2$

(**19**) $2x+y=-1$

(**20**) $x-2y=-12$

(**21**) $2x+y=-9$

(**22**) $5x+2y=33$

Answers to Exercise 2.3

(**1**) We need only compute the slope using two pairs of points on the graph. Then note that the slope using one pair does not equal the slope using the other pair.

When

$$\begin{aligned} x &= 0, \quad y = m0^2 + b = b \\ x &= 1, \quad y = m1^2 + b = m + b \\ x &= 2, \quad y = m2^2 + b = 4m + b. \end{aligned}$$

So points $P(0,b), Q(1,m+b), R(2,2m+b)$ are on the graph. We find the slope using two pairs of points.

$$\text{Using points P and Q} \implies \text{slope} = \frac{(m+b)-b}{1-0} = m$$

$$\text{Using points P and R} \implies \text{slope} = \frac{(4m+b)-b}{2-0} = 2m$$

Note that the slope using points P and Q does not equal the slope using points P and R. Conclusion: the graph of $y = mx^2 + b$, m is a constant not equal to 0, is not a straight line.

(**2**) Postponed.

(**3**) Let $y = mx^2 + b$ where m and b are constants. Suppose $P(x_1, y_1)$ and $Q(x_2, y_2)$ are any two points on the graph where $x_1 \neq x_2$ and x_1, x_2 are not both zero.

$$\text{slope using points } P \text{ and } Q = \frac{y_2 - y_1}{x_2 - x_1}.$$

Since, $y_1 = mx_1 + b$ and $y_2 = mx_2 + b$, we substitute

$$\frac{(mx_2^2 + b) - (mx_1^2 + b)}{x_2 - x_1} = m\frac{x_2^2 - x_1^2}{x_2 - x_1}.$$

$$\text{slope using points } P \text{ and } Q = m\frac{x_2^2 - x_1^2}{x_2 - x_1}.$$

Stuck. We cannot be sure that $m\dfrac{x_2^2 - x_1^2}{x_2 - x_1}$ has the same value for every selection of numbers x_1 and x_2.

(**4**) m in $y = mx^2 + b$ is not slope.

(**5**) Since the real numbers are closed under multiplication and subtraction, $y_1 - mx_1$ is a real number. We name it b.

(**6**) Yes. To see why, let $y = m_1x + b_1$ and $y = m_2x + b_2$ be two linear functions. Then the sum $(m_1x + b_1) + (m_2x + b_2) = (m_1 + m_2)x + (b_1 + b_2)$. Now

$m_1 + m_2$ is a constant, call it m and $b_1 + b_2$ is a constant, call it b. But $y = mx + b$, m and b constant, is a linear function.

(**7**) The theorem limits itself to non-vertical lines because the slope of a vertical line would be $\frac{y_2 - y_1}{0}$ which is nonsense.

(**8**)

(**a**) $y = x$ is the form $y = mx + b$ when $m = 1$ and $b = 0$. So by Theorem 2.1), its graph is a straight line.

(**b**) $y = x^{1.01}$ is not of the form $y = mx + b$ because the exponent on x is 1.01 not 1. Therefore, by Theorem 2.1 on page 23, its graph is not a straight line.

Answers to Exercise 3.1

(**1**) 5

(**2**) 11

(**3**) 8

(**4**) 6

(**5**) 10

(**6**) 100

(**7**) 13

(**8**) undefined

(**9**) $\frac{2}{5}$

(**10**) $\frac{8}{11}$

(**11**) $\frac{5}{9}$

(**12**) $\frac{12}{3}$

(**13**) 0.6

(**14**) 0.7

(**15**) 0.2

(**16**) 0.3

(**17**) 0.1

(**18**) 0.01

(**19**) 0.06

(**20**) $1, -2$

(**21**) $5, -5$

(**22**) $9, -9$

(**23**) $12, -12$

(**24**) $7, -7$

(**25**) $\frac{3}{7}, \frac{-3}{7}$

(**26**) $\frac{8}{5}, \frac{-8}{5}$

(**27**) 42

(**28**) 35

(**29**) 44

(**30**) 91

(**31**) 45

(**32**) 125

(**33**) 16

(**34**) 27

Answers to Exercise 3.2

(**1**) $2\sqrt{3}$

(**2**) $3\sqrt{5}$

(**3**) $5\sqrt{7}$

(**4**) $4\sqrt{6}$

(**5**) $10\sqrt{10}$

(**6**) $14\sqrt{7}$

(**7**) $6\sqrt{6}$

(**8**) $2\sqrt{2}$

(**9**) $24\sqrt{3}$

(**10**) $3\sqrt{3}$

(**11**) $2\sqrt{5}$

(**12**) $5\sqrt{10}$

(**13**) $7\sqrt{10}$

(**14**) 700

(**15**) 400

(**16**) $6\sqrt{14}$

Answers to Exercise 3.3

(**1**) $4\sqrt{3}$
(**2**) $4\sqrt{6}$
(**3**) $5\sqrt{2}$
(**4**) $5\sqrt{3}$
(**5**) $4\sqrt{6}$
(**6**) $3\sqrt{5}$
(**7**) $-9\sqrt{2}$
(**8**) $-\sqrt{3}$
(**9**) $-5\sqrt{6}$
(**10**) $7\sqrt{10}$
(**11**) $2\sqrt{6}$
(**12**) $-21\sqrt{2}$
(**13**) $25\sqrt{3}$
(**14**) undefined
(**15**) $-26\sqrt{5}$
(**16**) $11\sqrt{10}$
(**17**) $7\sqrt{5}+2\sqrt{6}$
(**18**) $-2\sqrt{5}-12\sqrt{6}$
(**19**) $-5\sqrt{7}$
(**20**) $15\sqrt{7}-6\sqrt{2}$

Answers to Exercise 3.4

(**1**) 5
(**2**) $-5\sqrt{2}$
(**3**) 40
(**4**) 12
(**5**) 36
(**6**) $54\sqrt{2}$
(**7**) $90\sqrt{2}$
(**8**) $-10\sqrt{30}$
(**9**) $5\sqrt{14}$
(**10**) $\frac{2}{3}\sqrt{5}$
(**11**) $3\sqrt{\frac{2}{7}}$
(**12**) undefined
(**13**) $70\sqrt{10}$
(**14**) $25\sqrt{21}$
(**15**) 30
(**16**) $30\sqrt{6}$
(**17**) $20\sqrt{7}$
(**18**) $2\sqrt{2}$
(**19**) $3\sqrt{7}$
(**20**) 1

Answers to Exercise 3.5

[Part 1]

(**1**) $\frac{\sqrt{10}}{5}$
(**2**) $\frac{\sqrt{6}}{2}$
(**3**) $\frac{\sqrt{7}}{7}$
(**4**) $\frac{3\sqrt{35}}{35}$
(**5**) $\frac{2\sqrt{15}}{5}$
(**6**) 2
(**7**) $\sqrt{6}$
(**8**) 1
(**9**) $\frac{\sqrt{35}}{21}$
(**10**) $\frac{\sqrt{2}}{2}$
(**11**) $\frac{5\sqrt{3}}{3}$
(**12**) undefined

[Part 2]

(**1**) $\frac{\sqrt{2}}{2}$ (**2**) $\sqrt{6}$ (**3**) $\sqrt{2}$ (**4**) $\sqrt{3}$ (**5**) $\sqrt{2}$ (**6**) $2\sqrt{2}$

Answers to Exercise 3.6

(**1**) $7\sqrt{6}$

(**2**) $4\sqrt{5}$

(**3**) $8\sqrt{3}$

(**4**) 12

(**5**) $-9+3\sqrt{6}$

(**6**) 7

(**7**) $15-3\sqrt{5}$

(**8**) $6+2\sqrt{3}$

(**9**) $5\sqrt{7}$

(**10**) $\frac{3\sqrt{2}-2\sqrt{3}}{6}$

(**11**) $\frac{-11\sqrt{2}}{2}$

(**12**) $\frac{15\sqrt{2}+10\sqrt{3}-6\sqrt{5}}{30}$

(**13**) 1

(**14**) $\frac{\sqrt{7}-1}{2}$

(**15**) $2\sqrt{2}$

(**16**) $\frac{26\sqrt{5}}{15}$

(**17**) $\frac{5\sqrt{2}}{12}$

(**18**) $4\sqrt{3}$

(**19**) $-\frac{10\sqrt{21}}{21}$

(**20**) $\frac{4\sqrt{6}-\sqrt{2}}{4}$

(**21**) If the product of two numbers is positive, the numbers must either be both negative or both positive. If one number is positive, the other must be positive, too.

(**22**)

$$\begin{aligned}(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b}) &= (\sqrt{a}-\sqrt{b})\sqrt{a}+(\sqrt{a}-\sqrt{b})\sqrt{b} \\ &= \sqrt{a}\sqrt{a}-\sqrt{a}\sqrt{b}+\sqrt{a}\sqrt{b}-\sqrt{b}\sqrt{b} \\ &= \sqrt{a}\sqrt{a}-\sqrt{b}\sqrt{b} \\ &= a-b.\end{aligned}$$

(**23**) Let a and b be two positive real numbers. Suppose that $\sqrt{a} > \sqrt{b}$. Then,

$$\begin{aligned}\sqrt{a}-\sqrt{b} &> 0. \\ (\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b}) &> 0 \\ a-b &> 0.\end{aligned}$$

(**24**) 17.89 (**25**) 56.57 (**26**) 178.9 (**27**) 0.5657 (**28**) 0.1789

Answers to Exercise 3.7

(**1**) (**a**) 1.414 (**b**) 2.449 (**c**) 2.828 (**d**) 3.162

Answers to Exercise 3.8

(**1**) $20\sqrt[3]{3}$

(**2**) $6\sqrt[4]{3}$

(**3**) $20\sqrt[3]{4}$

(**4**) $-8\sqrt[4]{2}$

(**5**) $-15\sqrt[3]{3}$

(**6**) $5\sqrt[4]{2}$

(**7**) $20\sqrt[3]{4}$

(**8**) $24\sqrt[4]{6}$

(**9**) $16\sqrt[3]{3}$

(**10**) $6\sqrt[5]{7}$

Answers to Exercise 3.9

(**1**) $-4\sqrt{5}-2\sqrt{3}$

(**2**) $-\sqrt{2}$

(**3**) $-2\sqrt[3]{4}$

(**4**) $-2\sqrt{3}+\sqrt{2}$

(**5**) $2\sqrt[4]{2}$

(**6**) $\sqrt{5}+2\sqrt{3}$

(**7**) $-2\sqrt[4]{3}-2\sqrt[4]{4}$

(**8**) $2\sqrt{3}$

(**9**) 0

(**10**) $6\sqrt{3}$

(**11**) $2\sqrt[3]{3}$

(**12**) $\sqrt[3]{4}-2\sqrt[3]{2}$

(**13**) $4\sqrt[4]{4}$

(**14**) $-4\sqrt[3]{2}-\sqrt[4]{2}$

(**15**) $-4\sqrt[4]{4}+2\sqrt[4]{2}$

(**16**) $-8\sqrt[3]{3}$

(**17**) $-2\sqrt[3]{3}$

(**18**) $-\sqrt[3]{4}+2\sqrt[4]{4}-2\sqrt[4]{3}$

(**19**) $-8\sqrt[3]{2}+2\sqrt[3]{4}$

(**20**) $-\sqrt[3]{2}$

(**21**) $-2\sqrt[3]{3}$

(**22**) $6\sqrt[3]{4}+4\sqrt[4]{2}$

(**23**) $-\sqrt[3]{4}-4\sqrt[4]{3}$

(**24**) $2\sqrt[4]{2}-2\sqrt[4]{4}$

(**25**) $-4\sqrt[3]{2}$

(**26**) $4\sqrt[3]{4}-2\sqrt[3]{2}$

(**27**) $-2\sqrt[3]{4}$

(**28**) $4\sqrt[3]{2}$

(**29**) $-5\sqrt[4]{3}-4\sqrt[4]{4}$

(**30**) $-3\sqrt[4]{4}-2\sqrt[4]{2}$

(**31**) $-6\sqrt[3]{6}$

(**32**) $-18\sqrt[4]{12}$

(**33**) $4\sqrt[4]{20}$

(**34**) $-24\sqrt[6]{18}$

(**35**) $2\sqrt[4]{5}$

(**36**) -90

(**37**) $3\sqrt[4]{10}$

(**38**) $2\sqrt[3]{36}$

(**39**) $6\sqrt[4]{2}$

(**40**) $-20\sqrt[4]{3}$

(**41**) $\frac{\sqrt[4]{12}}{8}$

(**42**) $-\frac{\sqrt[3]{12}}{2}$

(**43**) $-\frac{\sqrt[3]{3}}{6}$

(**44**) $\frac{\sqrt[3]{18}}{12}$

(**45**) $-\frac{\sqrt[3]{6}}{4}$

(**46**) $\frac{\sqrt[4]{9}}{3}$

(**47**) $\frac{\sqrt[6]{32}}{6}$

(**48**) $\frac{\sqrt[4]{27}}{3}$

(**49**) $\frac{\sqrt[4]{8}}{D}$

(**50**) $\frac{\sqrt[3]{6}}{2}$

Answers to Exercise 3.10

(**1**) (a) $\{2,3,5,7\}$ (b) non-prime.

(**2**) (a) $\{2,3,5,7\}$ (b) prime.

(**3**) (a) $\{2,3,5,7\}$ (b) prime.

(**4**) (a) $\{2,3,5,7,11,13\}$ (b) prime

(**5**) (a) $\{2,3,5,7,11,13,17,19\}$ (b) prime.

(**6**) (a) $\{2,3,5,7,11,13,17,19,23\}$ (b) non-prime

(**7**) (a) $\{2,3,5,7,11,13,17,19,23\}$ (b) prime.

(**8**) (a) $\{2,3,5,7,11\}$ (b) prime.

(**9**) (a) $\{2,3,5,7,11\}$ (b) prime.

(**10**) (a) $\{2,3,5,7,11,13\}$ (b) non-prime.

(**11**) (a) $\{2,3,5,7,11,13\}$ (b) non-prime.

(**12**) (a) $\{2,3,5,7,11,13\}$ (b) prime.

Answers to Exercise 4.1

(**1**) -9

(**2**) 9

(**3**) -27

(**4**) -27

(**5**) $2^5 = 32$

(**6**) $7^3 = 343$

(**7**) $2^7 = 128$

(**8**) $4 \cdot 9 = 36$

(**9**) x^9

(**10**) y^5

(**11**) x^{15}

(**12**) x^3

(**13**) Simplified

(**14**) 8100000

(**15**) $2^6 = 64$

(**16**) $8x^7y^5$

(**17**) $-8 \cdot 4 = -32$

(**18**) $-2^6 = -64$

(**19**) $-3^4 \cdot 2^2 = -81 \cdot 4 = -324$

(**20**) -1

(**21**) 1

(**22**) $8x^7$

(**23**) $4x^2 \cdot 8y^3 = 32x^2y^3$

(**24**) $2x^5 \cdot 4x^2 = 8x^7$

(**25**) $4y^2(-8)y^3 \cdot 3 = -96y^5$

(**26**) $\frac{8}{27}$

Answers to Exercise 4.2

(**1**) m^{12}

(**2**) 1

(**3**) $9x^4$

(**4**) $16n^6$

(**5**) $4v^6$

(**6**) $x^{12}y^9$

(**7**) $16u^8v^{12}$

(**8**) $a^{12}b^6$

(**9**) $16v^2u^8$ (**10**) $9y^8$

(**11**) 1

(**12**) $4k^4$

(**13**) $8a^9$

(**14**) $16a^{12}$

(**15**) $81n^4$

(**16**) $-16p^8$

(**17**) $-27x^3y^9$

(**18**) $81y^{16}$

(**19**) $-y^6$

(**20**) $16v^2u^8$

(**21**) x^8

(**22**) $-64x^9y^6$

(**23**) $\frac{16}{81}$

(**24**) $\frac{a^5}{32}$

(**25**) $\frac{a^6}{b^2}$

(**26**) $\frac{u^9}{v^{1}5}$

(**27**) $\frac{r^6}{t^4}$

(**28**) $\frac{-u^{15}}{v^{18}}$

(**29**) $\frac{-1}{v^{22}}$

(**30**) $\frac{x^6}{y^9}$

(**31**) 1

Answers to Exercise 4.3

(1) $\frac{1}{n^2}$ **(2)** $\frac{1}{3x^5}$ **(3)** $\frac{3}{4b^6}$ **(4)** $\frac{2}{p^3}$ **(5)** $\frac{1}{r^4}$ **(6)** $\frac{1}{2k}$

(7) $\frac{1}{4b^2}$ **(8)** $\frac{1}{n^2}$ **(9)** $4x^6$ **(10)** $\frac{3}{p^3}$ **(11)** $\frac{3}{5x^9}$ **(12)** $\frac{x}{3}$ **(13)** $\frac{4}{v^9}$

(14) $\frac{1}{p^4}$ **(15)** $\frac{4n^2}{5}$ **(16)** 3 **(17)** n^6 **(18)** $\frac{3x^3}{4}$ **(19)** $\frac{3n^3}{4}$ **(20)** $\frac{5}{4m^6}$

(21) $\frac{1}{r^4}$ **(22)** $\frac{1}{4x^5}$ **(23)** $\frac{3}{5x^5}$ **(24)** $\frac{2}{b^2}$ **(25)** $\frac{1}{k}$ **(26)** $\frac{1}{2k^8}$

(27) $1 \cdot a^m = a^{m+0} \implies a^0 \cdot a^m = a^{m+0}$

Answers to Exercise 4.4

(1) $\frac{1}{x^{30}}$ **(2)** r^{16} **(3)** x^{85} **(4)** p^{24} **(5)** $\frac{1}{b^{55}}$

(6) $\frac{1}{n^{16}}$ **(7)** 1 **(8)** $\frac{1}{m^5}$ **(9)** x^{75} **(10)** $\frac{1}{v^{44}}$

(11) k^{35} **(12)** a^{24} **(13)** $\frac{1}{v^{11}}$ **(14)** $-x^{23}$ **(15)** $-x^5$ **(16)** $-p^2$

(17) $-k^{12}$ **(18)** $-\frac{1}{x^{25}}$ **(19)** $\frac{1}{n^{360}}$ **(20)** $\frac{1}{n^5}$

Answers to Exercise 4.5

[Part 1]

(1) $\sqrt[3]{7}$ **(2)** $\sqrt[3]{25}$ **(3)** $(\sqrt{10})^3$ **(4)** $(\sqrt[3]{7})^4$

(5) $\sqrt{6}$ **(6)** $(\sqrt[3]{5})^4$ **(7)** $\sqrt{6b}$ **(8)** $(\sqrt[6]{x})^7$

(9) $(\sqrt[3]{6k})^4$ **(10)** $(\sqrt{5x})^3$ **(11)** $(\sqrt[3]{m})^4$ **(12)** $(\sqrt{6r})^3$

[Part 2]

(**1**) $10^{7/6}$
(**2**) $3^{2/3}$
(**3**) $7^{2/3}$
(**4**) $3^{7/4}$
(**5**) $4^{5/3}$
(**6**) $6^{5/2}$
(**7**) $(2r)^{2/3}$
(**8**) $(3m)^{5/2}$
(**9**) $(10x)^{3/4}$
(**10**) $(3n)^{7/4}$
(**11**) $(3k)^{5/4}$
(**12**) $(2p)^{3/4}$

[Part 3]

(**1**) $\frac{1}{(\sqrt[6]{10b})^7}$
(**2**) $\frac{1}{(\sqrt[3]{4n})^5}$
(**3**) $(\sqrt{x})^3$
(**4**) $\frac{1}{\sqrt[6]{2x}}$
(**5**) $\frac{1}{\sqrt[3]{4x}}$
(**6**) $\frac{1}{(\sqrt[3]{4v})^2}$

[Part 4]

(**1**) $(3r)^{-2/3}$
(**2**) $(x)^{-5/2}$
(**3**) $(7r)^{-4/3}$
(**4**) $(n)^{-5/6}$
(**5**) $(6v^2)^{-1/3}$
(**6**) $(5n)^{5/4}$

Answers to Exercise 4.6

(**1**) 3
(**2**) $\frac{1}{5}$
(**3**) 16
(**4**) 25
(**5**) $\frac{1}{3}$
(**6**) 1
(**7**) undefined
(**8**) 32
(**9**) 3
(**10**) $\frac{1}{4}$
(**11**) 1000
(**12**) $\frac{1}{25}$
(**13**) 32
(**14**) x^7
(**15**) x^{11}
(**16**) -2
(**17**) $\frac{1}{x}$
(**18**) x
(**19**) $x^{6/5}$
(**20**) $x^{7/12}$
(**21**) $\frac{x}{3}$
(**22**) x^{25}
(**23**) 1
(**24**) $x+1$
(**25**) $\frac{x}{9}$
(**26**) x^5
(**27**) $\frac{1}{x}$

Answers to Exercise 4.7

(**1**) To produce a counter example, choose $x = 3$ and $y = 4$. Then

$$\begin{aligned} \text{LHS} &= (x^2+y^2)^{-1/2} \\ &= (9+16)^{-1/2} \end{aligned}$$

$$
\begin{aligned}
&= (25)^{-1/2} \\
&= \frac{1}{5}.
\end{aligned}
$$

But,

$$
\begin{aligned}
\text{RHS} &= x^{-1} + x^{-1} \\
&= \frac{1}{3} + \frac{1}{4} \\
&= \frac{7}{12}.
\end{aligned}
$$

Note that LHS $\neq$ RHS.

(**2**)

$$
\begin{aligned}
\frac{x^{3/2} - x^{-3/2}}{x^{-3/2}} &= \frac{x^{3/2} - x^{-3/2}}{x^{-3/2}} \cdot \frac{x^{3/2}}{x^{3/2}} \\
&= x^{3/2}(x^{3/2} - x^{-3/2}) \\
&= x^{6/2} - x^{0} \\
&= x^{3} - 1.
\end{aligned}
$$

(**3**)

$$
\begin{aligned}
2^{-n} \cdot 8^{n-1} \cdot 4^{n+3} \div 16^{n} &= 2^{-n} \cdot (2^{3})^{n-1} \cdot (2^{2})^{n+3} \cdot (2^{-4})^{n} \\
&= 2^{-n} \cdot 2^{3(n-1)} \cdot 2^{2(n+3)} \cdot 2^{-4n} \\
&= 2^{-n} \cdot 2^{3n-3} \cdot 2^{2n+6} \cdot 2^{-4n} \\
&= 2^{-n+3n-3+2n+6-4n} \\
&= 2^{3} \\
&= 8.
\end{aligned}
$$

(**4**)

$$
\begin{aligned}
\left(\frac{10^{n+2}}{100}\right)^{1/n} &= \left(\frac{10^{n+2}}{10^{2}}\right)^{1/n} \\
&= (10^{n})^{1/n} \\
&= 10.
\end{aligned}
$$

(**5**)

$$
2^{x} \cdot 2 = 2^{x+1}.
$$

(**6**)

$$\begin{aligned} 2^{x+2} - (2^{x+1} + 2^x) &= 4 \cdot 2^x - (2 \cdot 2^x + 2^x) \\ &= 4 \cdot 2^x - 3 \cdot 2^x \\ &= 2^x. \end{aligned}$$

(**7**) Suppose $b \neq 0$ and a^n and b^n are both real numbers. Then

$$\begin{aligned} \text{LHS} &= \left(\frac{a}{b}\right)^n \\ &= \left(a \cdot \frac{1}{b}\right)^n \\ &= a^n \cdot \left(\frac{1}{b}\right)^n \\ &= a^n \cdot \frac{1}{b^n} \\ &= \frac{a^n}{b^n} \end{aligned}$$

(**8**)

Part a

$$\text{LHS} = (a^0)^n = 1^n = 1 = a^0 = a^{0 \cdot n} = \text{RHS}.$$

Part b

$$\text{LHS} = (ab)^0 = 1 = 1 \cdot 1 = a^0 b^0 = \text{RHS}.$$

Part c

$$\text{LHS} = \left(\frac{a}{b}\right)^0 = 1 = \frac{1}{1} = \frac{a^0}{b^0} = \text{RHS}.$$

(**9**) Suppose $a \neq 0$ and a^m and a^n are both real numbers. Then,

$$\begin{aligned} \text{LHS} &= \frac{a^m}{a^n} \\ &= a^m \cdot \frac{1}{a^n} \\ &= a^m \cdot a^{-n} \\ &= a^{m-n}. \end{aligned}$$

(**10**) $10^9 = (2 \cdot 5)^9 = 2^9 \cdot 5^9$. Then $10 \cdot 10 = 100$. There are 100 factors of 1 billion. See Section 1.6.1 on page 17.

Answers to Exercise 5.1

(**1**) $10x^2+27x+5$
(**2**) $k^2+4k-21$
(**3**) $6x^2+8x+2$
(**4**) $56n^2+76n+24$
(**5**) $24x^2-24x+6$
(**6**) $49m^2+105m+56$
(**7**) $14v^2-40v+24$
(**8**) $12x^2+4x-56$
(**9**) $48n^2-10n-2$
(**10**) $16n^2-68n+42$
(**11**) $24n^2-6$
(**12**) $6x^2+23x+7$
(**13**) $18v^2-42v-16$
(**14**) $7p^2+37p-30$
(**15**) $42x^2+47x+10$
(**16**) $5r^2+24r+16$
(**17**) $30x^2+24x-6$
(**18**) $21r^2-60r+36$
(**19**) $4n^2+20n+16$
(**20**) $12x^2-23x+5$
(**21**) $18x^2+3x-21$
(**22**) $12r^2+32r+16$
(**23**) $12n^2-40n-7$
(**24**) $3b^2+9b-12$
(**25**) $30n^2-12n-18$
(**26**) $32n^2+32n+8$
(**27**) $7k^2+50k-48$
(**28**) $40n^2-21n-49$
(**29**) $9a^2+6a-24$
(**30**) $40x^2-2x-2$

Answers to Exercise 5.2

(**1**) $(x+6)(x-8)$
(**2**) $(p+10)(p-9)$
(**3**) $(x+10)(x+5)$
(**4**) $(n-1)(n+9)$
(**5**) $(b+2)(b-10)$
(**6**) $(p+3)(p+1)$
(**7**) $(b-1)(b-6)$
(**8**) $(a+7)(a-6)$
(**9**) $(k-7)(k+4)$
(**10**) $(v-2)(v-3)$
(**11**) $(x+8)(x-10)$
(**12**) $(x+6)(x+7)$
(**13**) $(n+7)(n+10)$
(**14**) $(n+10)(n+6)$
(**15**) $(x-2)(x+3)$
(**16**) $(x-4)(x+5)$
(**17**) $(r-10)(r-3)$
(**18**) $(x+6)(x+8)$
(**19**) $(x-8)(x+10)$
(**20**) $(x+4)(x-2)$
(**21**) $(a-9)(a-8)$
(**22**) $(a-3)(a+8)$
(**23**) $(r+4)(r+3)$
(**24**) $(n+9)(n-5)$
(**25**) $(x-10)(x+2)$
(**26**) $(x+9)(x+1)$
(**27**) $(n-9)(n+5)$
(**28**) $(n-5)(n-9)$
(**29**) $(n-4)(n-3)$
(**30**) $(p+1)(p-5)$
(**31**) $(v+3)(v-5)$
(**32**) $(b-10)(b+5)$
(**33**) $(x+1)(x-3)$
(**34**) $(b+5)(b-4)$
(**35**) $(n-1)(n-7)$
(**36**) $(k-1)(k+5)$
(**37**) $(r+6)(r-1)$
(**38**) $(v-8)(v+3)$
(**39**) $(x+1)(x+7)$
(**40**) $(n+9)(n-3)$
(**41**) $(n-4)(n-10)$
(**42**) $(k+5)(k-7)$
(**43**) $(k+2)(k+5)$
(**44**) $(m-6)(m-4)$
(**45**) $(m-8)(m-2)$
(**46**) $(r+1)(r+5)$
(**47**) $(x-10)(x+9)$
(**48**) $(x+3)(x+8)$
(**49**) $(m-5)(m-6)$
(**50**) $(x+9)(x-4)$
(**51**) $(x+7)(x+6)$
(**52**) $(b-4)(b+2)$
(**53**) $(n-7)(n-2)$
(**54**) $(x-9)(x+7)$
(**55**) $(v-6)(v+1)$
(**56**) $(x-7)(x+5)$
(**57**) $(p-2)(p+6)$
(**58**) $(n+4)(n-9)$
(**59**) $(a-6)(a+10)$
(**60**) $(m-3)(m-8)$

Answers to Exercise 5.3

(1) $(3m-4)(m-7)$
(2) $(3x-5)(x-3)$
(3) $(5n+6)(n-5)$
(4) $(7b+6)(b+10)$
(5) $(7x-5)(x+4)$
(6) $(3p-7)(p+5)$
(7) $(5n-3)(n+4)$
(8) $(7r+9)(r+7)$
(9) $(7m+10)(m-8)$
(10) $(2n+1)(n-2)$
(11) $(3x+10)(x+4)$
(12) $(7x-1)(x+5)$
(13) $(5n-2)(n+9)$
(14) $(7a+6)(a-5)$
(15) $(3v+4)(v-6)$
(16) $(3r+8)(r-9)$
(17) $(7k-6)(k+10)$
(18) $(3x-10)(x+10)$
(19) $(3k-5)(k-8)$
(20) $(3x+2)(x-4)$
(21) $(5n-8)(n+1)$
(22) $(5p-2)(p-6)$
(23) $(7v+9)(v+2)$
(24) $(3x-10)(x+8)$
(25) $(5b-2)(b+10)$
(26) $(5n-8)(n-6)$
(27) $(5m+1)(m+7)$
(28) $(3v-1)(v-1)$
(29) $(5p-8)(p+9)$
(30) $(3m+10)(m+2)$
(31) $(7m+8)(m-4)$
(32) $(7r-10)(r+1)$
(33) $(2r-3)(r-7)$
(34) $(7x+8)(x-10)$
(35) $(3b+2)(b+7)$
(36) $(5b+7)(b+6)$
(37) $(3v+10)(v-3)$
(38) $(7p+4)(p+3)$
(39) $(7n+5)(n+8)$
(40) $(5x+2)(x-9)$
(41) $(7n-4)(n+8)$
(42) $(7m-9)(m-5)$
(43) $(3n-7)(n-4)$
(44) $(2p-9)(p-2)$
(45) $(7a-10)(a-7)$
(46) $(2v+3)(v-2)$
(47) $(2x+7)(x+1)$
(48) $(2x-5)(x+6)$
(49) $(2x+3)(x-9)$
(50) $(5x-7)(x-9)$
(51) $(7x-6)(x+7)$
(52) $(2n+1)(n+6)$
(53) $(2r-5)(r-8)$
(54) $(2x-9)(x+4)$
(55) $(3v+5)(v+5)$
(56) $(7n+3)(n-7)$
(57) $(5v+4)(v+10)$
(58) $(7n+3)(n+7)$
(59) $(7v-4)(v+3)$
(60) $(7p-3)(p-1)$

Answers to Exercise 5.4

(1) $(3a-4)(b+5)$
(2) $(4x+p)(4y+p)$
(3) $(x-5)(3y+4)$
(4) $(5u+1)(v+5)$
(5) $(4x-3)(2y+3x)$
(6) $(5x-3)(2y+5x)$
(7) $(5u-3v)(4v+5)$
(8) $(u-3)(4v-1)$
(9) $(2a+3n)(3b-4a)$
(10) $(4x+3n)(3y-1)$
(11) $(b-2x)(5u-4v)$
(12) $(5m-2n)(3u-v)$
(13) $(5x+4p)(4y-1)$
(14) $(3a-2b^2)(w-3k)$
(15) $(a+r)(2b+a)$
(16) $(4x+3)(4y-5)$
(17) $(3x+4k)(5y-2)$
(18) $(3u-5x)(5v-3u^3)$
(19) $(3x+2v)(y-2x^2)$
(20) $(5a+2y)(4z+3c)$
(21) $(8x+3n)(3y+4n)$
(22) $(m-5)(6n+7)$
(23) $(2x-1)(5y+3)$
(24) $(x-8y^2)(7y+3)$
(25) $(2m-7)(2n-5m)$
(26) $(7x+8y)(y-6)$
(27) $(7u-5x)(3v+5u^2)$
(28) $(5x+6)(y+3)$
(29) $(4u-3)(2v+1)$
(30) $(5x-4)(y+8)$
(31) $(x+1)(2y-3)$
(32) $(5x+3b)(5y-9x^3)$
(33) $(2x+1)(5y-3)$
(34) $(8a+5b)(9h-2k^2)$
(35) $(9x-7a)(3y+2)$
(36) $(5p^2+q)(w^2+10k)$
(37) $(a-2x)(7b-a)$
(38) $(5m+6)(7n+3)$
(39) $(u-x)(6v+1)$
(40) $(8x+3p)(7y-5p)$
(41) $(7u-2r)(10v+r)$
(42) $(5x-y)(w-3k)$
(43) $(7x-8)(2y-7)$
(44) $(5a-x^2)(10z+9c)$
(45) $(9x+7n)(y-10)$
(46) $(x+7n)(3y+8)$
(47) $(3x+p)(6y-1)$
(48) $(a+8x)(z+4c)$
(49) $(5u+8x)(4v-9)$
(50) $(2x+1)(10y-7)$
(51) $(7x+1)(2y+5)$
(52) $(3u+5)(8v+1)$
(53) $(x+n)(7y+3x^2)$
(54) $(9m^2+5n)(10h+9k)$
(55) $(6b+5x)(5z-9c)$
(56) $(9a+4b)(7b+10)$

Answers to Exercise 5.5

(1) $(2n+1)(3n-4)$
(2) $(n-4)(4n+7)$
(3) $(2x+3)(3x+2)$
(4) $(2n+7)(3n+5)$
(5) $(a-6)(4a-3)$
(6) $(r-7)(6r-7)$
(7) $(2k+1)(3k+2)$
(8) $(x-2)(6x+5)$
(9) $(2n-1)(2n-5)$
(10) $(3r+1)(2r+5)$
(11) $(2m-5)(3m+7)$
(12) $(x+6)(4x+1)$
(13) $(n+6)(4n+3)$
(14) $(2p+7)(3p+4)$
(15) $(2n+5)(2n+3)$
(16) $(m-6)(6m+7)$
(17) $(n-5)(6n+5)$
(18) $(2b+3)(3b+4)$
(19) $(2b-7)(3b+2)$
(20) $(2v-5)(3v+4)$
(21) $(2v+1)(3v-2)$
(22) $(2x+7)(3x-4)$
(23) $(n-3)(6n-1)$
(24) $(v+3)(4v+3)$
(25) $(3b+2)(2b+1)$
(26) $(a-4)(4a-7)$
(27) $(2n+7)(3n-2)$
(28) $(x-4)(6x-1)$
(29) $(2a+1)(3a+7)$
(30) $(2v+3)(3v-5)$
(31) $(r-7)(10r-1)$
(32) $(5n+4)(2n+9)$
(33) $(4x+5)(3x-2)$
(34) $(x-3)(9x-1)$
(35) $(b-10)(9b-11)$
(36) $(b-5)(9b-4)$
(37) $(x+11)(9x+7)$
(38) $(4x+7)(3x+8)$
(39) $(2x+9)(6x-7)$
(40) $(2p-3)(4p+11)$
(41) $(n-11)(9n-5)$
(42) $(m-7)(9m-7)$
(43) $(3m+2)(4m-5)$
(44) $(n+12)(10n+3)$
(45) $(m+6)(9m+2)$
(46) $(4x+5)(3x-10)$
(47) $(p+10)(10p-1)$
(48) $(r-5)(8r+3)$
(49) $(x+4)(9x+4)$
(50) $(2n-9)(3n-2)$
(51) $(3n-10)(4n+9)$
(52) $(3r-4)(3r-8)$
(53) $(5x+7)(2x-5)$
(54) $(n-8)(9n+10)$
(55) $(n+9)(9n+10)$
(56) $(4x-9)(3x-10)$
(57) $(m-2)(9m-11)$
(58) $(p-5)(9p-10)$
(59) $(x-9)(9x-4)$
(60) $(5x-8)(2x+3)$

Answers to Exercise 5.6

(**1**) $4(r-3)(r+8)$

(**2**) $3(n-6)(n+2)$

(**3**) $3(n+1)(n-5)$

(**4**) $4(x+10)(x-1)$

(**5**) $3(x+6)(x-10)$

(**6**) $4(v+7)(v+1)$

(**7**) $4(x+2)(x+5)$

(**8**) $2(b-2)(b-1)$

(**9**) $5(x-4)(x+3)$

(**10**) $3(n-10)(n-5)$

(**11**) $5(n+6)(n+5)$

(**12**) $5(r-8)(r-10)$

(**13**) $5(x-10)(x-3)$

(**14**) $3(r-1)(r+7)$

(**15**) $3(n+7)(n+2)$

(**16**) $5(n-5)(n-8)$

(**17**) $2(x-9)(x-7)$

(**18**) $3(x-8)(x-5)$

(**19**) $5(p-2)(p+4)$

(**20**) $6(m+9)(m+2)$

(**21**) $4(p+3)(p-7)$

(**22**) $2(m-7)(m-9)$

(**23**) $4(r+6)(r+2)$

(**24**) $6(m-5)(m+8)$

(**25**) $6(x+3)(x+9)$

(**26**) $2(x+8)(x+6)$

(**27**) $3(r+9)(r-4)$

(**28**) $3(n+5)(n+10)$

(**29**) $2(x+1)(x-8)$

(**30**) $6(a-3)(a+9)$

(**31**) $x(7x-9)(x-3)$

(**32**) $2x(7x+9)(x-1)$

(**33**) $x(5x-6)(x-4)$

(**34**) $2v^2(3v-2)(v+1)$

(**35**) $6x(5x+9)(x+5)$

(**36**) $3v(7v+8)(v-10)$

Answers to Exercise 5.7

(**1**) $(x+6)^2$
(**2**) $(5x+6)(5x-6)$
(**3**) $(4b-7)^2$
(**4**) $(2m-1)^2$
(**5**) $(5a-6)^2$
(**6**) $(8n-13)^2$
(**7**) $(3x+13)^2$
(**8**) $(7b+2)(7b-2)$
(**9**) $(14p+1)(14p-1)$
(**10**) $(7x+13)(7x-13)$
(**11**) $(7x-10)^2$
(**12**) $(5v+6)^2$
(**13**) $(5p-4)^2$
(**14**) $(5p+11)(5p-11)$
(**15**) $(3k+8)^2$
(**16**) $(2x+3)^2$
(**17**) $(7n+10)^2$
(**18**) $(9b+10)(9b-10)$
(**19**) $(10v-3)^2$
(**20**) $(3r-11)^2$
(**21**) $(4k-1)^2$
(**22**) $(4a+3)(4a-3)$
(**23**) $(4n-3)^2$
(**24**) $(x+5)(x-5)$
(**25**) $(3a+2)^2$
(**26**) $(5n+11)^2$
(**27**) $(5r+9)^2$
(**28**) $(11x+1)(11x-1)$
(**29**) $(8k+5)(8k-5)$
(**30**) $(10v+3)^2$
(**31**) $(9v+10)^2$
(**32**) $(4n-5)^2$
(**33**) $(9m+4)^2$
(**34**) $(7r+2)^2$
(**35**) $(5n+8)^2$
(**36**) $(13x+14)(13x-14)$
(**37**) $(2m+13)^2$
(**38**) $(6n-5)^2$
(**39**) $(10m+1)^2$
(**40**) $(8a+11)^2$
(**41**) $(12x+11)^2$
(**42**) $(9x+7)(9x-7)$
(**43**) $(8r+13)(8r-13)$
(**44**) $(7x+4)^2$
(**45**) $(13x+10)^2$
(**46**) $(7x-5)^2$
(**47**) $(6n+13)(6n-13)$
(**48**) $(11x+8)^2$
(**49**) $(2x+7)^2$
(**50**) $(8x-5)^2$
(**51**) $(13x-6)^2$
(**52**) $(9a-14)^2$
(**53**) $(r-11)^2$
(**54**) $(9m+1)^2$
(**55**) $(11b+12)(11b-12)$
(**56**) $(3b-1)^2$
(**57**) $(4x+5)^2$
(**58**) $(3n-14)^2$

Answers to Exercise 5.8

(1) $(x+2)(x+9)$
(2) $(x-2)(x+1)$
(3) $(y^2-3)(y^2+3)$
(4) $(x-2)(x+2)(x^2+4)$
(5) $(3a-2)(3a+2)(9a^2+4$
(6) $(2x-1)(2x+1)(4x^2+1$
(7) $(x-6)(x+4)$
(8) $(x-3y-7)(x+3y-7)$
(9) $(2x-4y+9)(2x+4y+9)$
(10) $(a-10b-8)(a+10b-8)$
(11) $(6a-11b-7)(6a+11b-7)$
(12) $(x-y-2z)(x+y+2z)$
(13) $(3x+2)(3x+8)$
(14) $(8x^2-4x+1)(8x^2+4x+1)$
(15) $(a-b)(a+b+1)$
(16) $(x-5)(3x-1)$
(17) $(x-y-2)(x+y+2)$
(18) $(x-9)(x-3)(x+3)$
(19) $(a-b)(a-b+1)$

Answers to Exercise 5.9

(1) $\frac{1}{6}(x-1)(x-3)$
(2) $\left(x+\frac{2}{3}\right)(x+1)$
(3) $\left(x-\frac{1}{4}\right)\left(x+\frac{1}{2}\right)$
(4) $\left(x+\frac{1}{2}\right)^2$
(5) $\left(x-\frac{1}{5}\right)\left(x+\frac{1}{5}\right)$
(6) $\frac{1}{3}\left(x+\frac{1}{2}\right)^2$
(7) $(x-\sqrt{2})(x+\sqrt{3}$
(8) $(x-\sqrt{5})(x-\sqrt{2})$
(9) $(x-\sqrt{2})(x-1)$
(10) $(x-\sqrt{2})^2$
(11) $(x+\sqrt{5})^2$
(12) $(x-\sqrt{5})(x+\sqrt{5})$
(13) $(x-\sqrt{7})(x+\sqrt{7})$
(14) $(x-\sqrt{11})(x+\sqrt{7})$
(15) $(x-\sqrt{2})(x-\sqrt{7})$
(16) $(x-2)(x-\sqrt{8})$

Answers to Exercise 6.1

(1) $\frac{3}{x-4}$

(2) $\frac{5x}{x+8}$

(3) $\frac{5v}{v+1}$

(4) $\frac{b-9}{b+5}$

(5) $\frac{k+9}{k-7}$

(6) $\frac{7-p}{p-3}$

(7) $\frac{k-1}{k+6}$

(8) $\frac{8-x}{x+5}$

(9) $\frac{p-6}{p+10}$

(10) $\frac{x-3}{x+9}$

(11) $-\frac{v+8}{8}$

(12) $\frac{b+7}{2b}$

(13) $\frac{(r-8)(r-4)}{3}$

(14) $4-x$

(15) $\frac{8}{x-3}$

(16) $\frac{x-3}{x+8}$

(17) $\frac{56m^2}{3}$

(18) $m-7$

(19) $\frac{5n^2(n-5)}{n-8}$

(20) $\frac{7}{k-3}$

Answers to Exercise 6.2

(1) $\frac{-5a+9}{5(5a-2)}$

(2) $\frac{3a^2-7a-22}{3(a+4)(a+2)}$

(3) $\frac{5n^2-25n+4}{2(n-5)}$

(4) $-\frac{b+18}{2(b-6)}$

(5) $\frac{3x^2-1}{x+1}$

(6) $\frac{5k^2-21k+8}{2(k-5)(k+2)}$

(7) $\frac{26k-22}{3k(5k-3)}$

(8) $\frac{7n-25}{(n-4)(n-3)}$

(9) $\frac{5r^2-9r}{(r-5)(5+3)}$

(10) $\frac{5b-14}{(b-4)(b-1)}$

(11) $\frac{3x^2-2x-12}{(x-3)(x-2)}$

(12) $-\frac{2m-7}{2m(m-3)}$

(13) $-\frac{4p+48}{5(p-3)}$

(14) $-\frac{2v^2-v+6}{(v-2)(v+1)}$

(15) $-\frac{6n^2-10n-15}{3(3n-5)}$

(16) $-\frac{6x^2+7x-16}{(3x-1)(x+4)}$

(17) $-\frac{10x+6}{3(x-3)}$

(18) $-\frac{2v-13}{v(v-6)}$

(19) $\frac{7n-30}{2(n-4)}$

(20) $-\frac{x^2-6x}{(x-3)(x-2)}$

Answers to Exercise 6.3

(1) 0

(2) $\dfrac{1}{1+p}$

(3) $\dfrac{n-2}{n(n-4)}$

(4) $\dfrac{16a^2-28a}{(5a-2)(a+5)}$

(5) $\dfrac{5b+23}{3(b+1)}$

(6) $\dfrac{18n-17}{2n(3n-2)}$

(7) $\dfrac{33a+27}{(a+1)(6a+5)}$

(8) $\dfrac{-4x^2+26x+30}{(x-5)(x+5)}$

(9) $\dfrac{4x^2+19x-9}{6x(x+4)}$

(10) $\dfrac{3r^3+21r^2+31r}{(r+4)(r+3)}$

(11) $\dfrac{-k}{(k+1)(3k+2)}$

(12) $\dfrac{-x^2+13x-4}{(x-5)(x-1)}$

(13) $\dfrac{6x^2+11x14}{2(x-4)(x+3)}$

(14) $\dfrac{7k^2+47k+90}{3k(k+5)(k+3)}$

(15) $\dfrac{2n^3-4n^2-48n+25}{5(n-6)(n+4)}$

(16) $\dfrac{9-3r}{(r-1)(r+2)}$

(17) $\dfrac{3x^2+50x+80}{2x(x+4)^2}$

(18) $-\dfrac{x^3+3x^2-20x+8}{2(x-3)(x+6)}$

(19) $\dfrac{21n-30}{(n-6)(3n-2)}$

(20) $\dfrac{3x^2-27x+55}{(x-6)(x3)}$

(21) $\dfrac{9x^2+36x+41}{3(x+2)^2}$

(22) $\dfrac{2p^3+8p^2-23p+4}{(p-2)(p+6)}$

(23) $\dfrac{n^2+4n+20}{(n-1)(n+4)}$

(24) $\dfrac{2r^3+6r^2+r-2}{r(r-2)(r+3)}$

(25) $\dfrac{15b^2+16b-6}{5b(b-6)(b+1)}$

(26) $\dfrac{4b^2-24b+1}{2(b-6)}$

(27) $\dfrac{5x^3+8x^2+10x-8}{5x^2(x-4)(x-1)}$

(28) $\dfrac{-5r^2-9r+121}{3(r-4)(r+6)}$

(29) $\dfrac{4n^2-5n-11}{5(n-3)(n+3)}$

(30) $\dfrac{3v^2-3v-1}{v-1}$

(31) $\dfrac{p^2-4p+12}{p(p-4)(p-3)}$

(32) $\dfrac{13}{2(x+5)}$

Answers to Exercise 6.4

(1) $\frac{x-6}{x-3}$

(2) $\frac{x^2-11x-6}{(x-3)(x+3)}$

(3) $\frac{2x-14}{3x+21}$

(4) $\frac{(x-7)(x+2)}{(x+1)(x+7)}$

(5) $\frac{x+7}{x+2}$

(6) $\frac{x-1}{x^2+9}$

(7) $-\frac{x-2}{x+1}$

(8) $-\frac{x-5}{x+2}$

(9) $\frac{2x+1}{x+7}$

(10) $\frac{(x-2)(x+3)}{(x-5)(x-3)}$

(11) $\frac{(x-2)(x^2+1)}{x-5}$

(12) $\frac{x+4}{x+7}$

(13) 1

(14) $\frac{(x+2)(x+6)}{(x-4)(x-2)}$

(15) $\frac{x-1}{x+9}$

(16) $\frac{x+5}{x+4}$

Answers to Exercise 7.1

(1) $x=0$ or $x=1$
(2) $n=-3$ or $n=-7$
(3) $x=-1$ or $x=2$
(4) $x=2$ or $x=-8$
(5) $a=2$ or $a=-6$
(6) $v=6$ or $v=1$
(7) $b=7$ or $b=-6$
(8) $x=6$ or $x=-2$
(9) $m=5$
(10) $x=-3$ or $x=8$
(11) $x=-2$ or $x=4$
(12) $x=-7$ or $x=3$
(13) $b=6$ or $b=7$
(14) $n=-7$ or $n=5$
(15) $m=6$ or $m=5$
(16) $x=-3$ or $x=-6$
(17) $n=2$ or $n=-8$
(18) $r=2$ or $r=8$
(19) $x=2$ or $x=-3$
(20) $x=-2$ or $x=2$
(21) $v=-7$ or $v=1$
(22) $r=-3$ or $r=7$
(23) $b=-3$ or $b=1$
(24) $p=6$ or $p=-4$
(25) $x=-7$ or $x=3$
(26) $n=6$ or $n=5$

(27) $k=-7$ or $k=-5$
(28) $a=6$ or $a=-6$
(29) $x=2$ or $x=7$
(30) $m=2$ or $m=6$
(31) $n=4/5$ or $n=4$
(32) $v=-8/7$ or $v=7/2$
(33) $x=-8/3$ or $x=-6$
(34) $b=-7/3$ or $b=6$
(35) $x=4/7$ or $x=8/3$
(36) $x=8/7$ or $x=0$
(37) $n=6/7$ or $n=6$
(38) $a=3/7$ or $a=-1$
(39) $p=-4/7$ or $p=-4$
(40) $k=-6/5$ or $k=-1$
(41) $m=-1/5$ or $m=-4/7$
(42) $n=-7/3$ or $n=-1$

Answers to Exercise 7.2

(**1**) $x = -3$ or $x = 3$

(**2**) $x = -5$ or $x = 5$

(**3**) $x = 6$ or $x = 9$

(**4**) $x = 5$

(**5**) $x = 1$ or $x = 2$

(**6**) $x = -3$

(**7**) $x = 3/2$

(**8**) $x = 1/4$ or $x = 3/2$

(**9**) $x = -1$ or $x = -9$

(**10**) $x = -1$

Answers to Exercise 7.3

(**1**) $x^2 + 14x + 49 = (x+7)^2$

(**2**) $x^2 - 20x + 100 = (x-10)^2$

(**3**) $x^2 - 16x + 64 = (x-8)^2$

(**4**) $x^2 + 7x + \frac{49}{4} = \left(x + \frac{7}{2}\right)^2$

(**5**) $x^2 - 9x + \frac{81}{4} = \left(x - \frac{9}{2}\right)^2$

(**6**) $x^2 + \frac{2x}{3}x + \frac{1}{9} = \left(x + \frac{1}{3}\right)^2$

(**7**) $x^2 + 5x + \frac{25}{4} = (x + \frac{5}{2})^2$

(**8**) $x^2 - 3x + \frac{9}{4} = (x - \frac{3}{2})^2$

(**9**) $x^2 - 6x + 9 = (x-3)^2$

(**10**) $x^2 - 8x + 16 = (x-4)^2$

(**11**) $x^2 + \frac{3}{5}x + \frac{9}{100} = (x + \frac{3}{10})^2$

(**12**) $x^2 + \frac{2}{5}x + \frac{1}{25} = (x + \frac{1}{5})^2$

(**13**) $x^2 - \frac{3}{7}x + \frac{9}{196} = (x - \frac{3}{14})^2$

(**14**) $x^2 - \frac{1}{4}x + \frac{1}{64} = (x - \frac{1}{8})^2$

(**15**) $x^2 - \frac{1}{3}x + \frac{1}{36} = (x - \frac{1}{6})^2$

(**16**) $x^2 + \frac{3}{2}x + \frac{9}{16} = (x + \frac{3}{4})^2$

Answers to Exercise 7.4

(**1**) $\left(n-2-\sqrt{3}\right)\left(n-2+\sqrt{3}\right)$

(**2**) $\left(m+2-\sqrt{3}\right)\left(m+2+\sqrt{3}\right)$

(**3**) $\left(n-1-\sqrt{2}\right)\left(n-1+\sqrt{2}\right)$

(**4**) $\left(p+3-2\sqrt{2}\right)\left(p+3+2\sqrt{2}\right)$

(**5**) $(x-9)(x+1)$

(**6**) $\left(m-\frac{-5+\sqrt{5}}{2}\right)\left(m-\frac{-5-\sqrt{5}}{2}\right)$

(**7**) $\left(n-\frac{7+5\sqrt{5}}{2}\right)\left(n-\frac{7-5\sqrt{5}}{2}\right)$

(**8**) $\left(a-\frac{-3+\sqrt{37}}{2}\right)\left(a-\frac{-3-\sqrt{37}}{2}\right)$

(**9**) $\left(x-\frac{1+\sqrt{69}}{2}\right)\left(x-\frac{1-\sqrt{69}}{2}\right)$

(**10**) $\left(x-\frac{-5+\sqrt{33}}{2}\right)\left(x-\frac{-5-\sqrt{33}}{2}\right)$

(**11**) $2(p-2)(p+6)$

(**12**) $4\left(x-\frac{3}{2}\right)\left(x-\frac{1}{2}\right)$

(**13**) $5\left(k-\frac{-5+\sqrt{35}}{5}\right)\left(k-\frac{-5-\sqrt{35}}{5}\right)$

(**14**) $5\left(a-\frac{-5+\sqrt{70}}{5}\right)\left(a-\frac{-5-\sqrt{70}}{5}\right)$

(**15**) $3\left(b-1-\sqrt{5}\right)\left(b-1+\sqrt{5}\right)$

(**16**) $3(m-4)\left(m+\frac{2}{3}\right)$

(**17**) $3\left(n-\frac{2+\sqrt{31}}{3}\right)\left(n-\frac{2-\sqrt{31}}{3}\right)$

(**18**) $2\left(m-\frac{1+\sqrt{43}}{2}\right)\left(m-\frac{1-\sqrt{43}}{2}\right)$

(**19**) $3\left(r-\frac{10}{3}\right)(r+2)$

(**20**) $5\left(b-\frac{-9+\sqrt{41}}{10}\right)9b-\left(\frac{-9-\sqrt{41}}{10}\right)$

(**21**) $4\left(x+\frac{1}{2}\right)(x+2)$

(**22**) $5\left(k-\frac{-3+3\sqrt{6}}{5}\right)\left(k-\frac{-3-3\sqrt{6}}{5}\right)$

(**23**) $2(m-2)\left(m+\frac{9}{2}\right)$

(**24**) $\left(b-\frac{1+\sqrt{41}}{2}\right)\left(b-\frac{1-\sqrt{41}}{2}\right)$

Answers to Exercise 7.5

(**1**) $\frac{-7\pm 3\sqrt{13}}{2}$

(**2**) $\frac{-1\pm\sqrt{77}}{2}$

(**3**) $\frac{-9\pm\sqrt{33}}{2}$

(**4**) $\frac{5\pm\sqrt{53}}{2}$

(**5**) $\frac{-3\pm\sqrt{37}}{2}$

(**6**) $\frac{1\pm\sqrt{93}}{2}$

(**7**) $\frac{-3\pm\sqrt{41}}{8}$

(**8**) No solution

(**9**) 3 or $-5/4$

(**10**) $\frac{-3\pm\sqrt{69}}{5}$

(**11**) $\frac{-1\pm\sqrt{181}}{2}$

(**12**) $\frac{3\pm\sqrt{21}}{2}$

(**13**) $\frac{-5\pm\sqrt{10}}{3}$

(**14**) $\frac{2\pm\sqrt{46}}{3}$

(**15**) $\frac{-3\pm\sqrt{21}}{2}$

(**16**) $\frac{1\pm 3\sqrt{5}}{4}$

(**17**) $\frac{-4\pm 2\sqrt{34}}{5}$

(**18**) $\frac{-3\pm\sqrt{57}}{8}$

(**19**) $\frac{-5\pm\sqrt{35}}{2}$

(**20**) $\frac{1\pm\sqrt{97}}{2}$

(**21**) -2 or -7

(**22**) $6/5$ or -2

(**23**) $\frac{-3\pm\sqrt{101}}{2}$

(**24**) $\frac{-5\pm\sqrt{13}}{2}$

(**25**) $\frac{-2\pm 2\sqrt{11}}{5}$

(**26**) $\frac{-3\pm\sqrt{65}}{2}$

(**27**) $\frac{-1\pm\sqrt{41}}{2}$

(**28**) $\frac{9\pm\sqrt{69}}{6}$

(**29**) $\frac{-1\pm\sqrt{17}}{4}$

(**30**) $\frac{5\pm\sqrt{69}}{2}$

(**31**) $\frac{5\pm\sqrt{61}}{2}$

(**32**) $\frac{-9\pm\sqrt{17}}{2}$

(**33**) $\frac{-5\pm 5\sqrt{5}}{2}$

(**34**) $\frac{9\pm 5\sqrt{5}}{2}$

(**35**) 1 or -8

(**36**) 2 or -5

(**37**) No solution

(**38**) $\frac{1\pm\sqrt{33}}{4}$

(**39**) 3 or $-3/4$

(**40**) 3 or -4

(**41**) 4 or -1

(**42**) $\frac{9\pm 3\sqrt{13}}{2}$

Answers to Exercise 7.6

(**1**) $\frac{-1\pm\sqrt{22}}{3}$

(**2**) $\frac{-3\pm2\sqrt{6}}{3}$

(**3**) $\frac{-6\pm\sqrt{102}}{3}$

(**4**) $-4\pm\sqrt{26}$

(**5**) $2\pm\sqrt{3}$

(**6**) $\frac{-5\pm\sqrt{29}}{4}$

(**7**) $\frac{5\pm3\sqrt{5}}{10}$

(**8**) $-10/3$ or 7

(**9**) $\frac{-5\pm\sqrt{55}}{3}$

(**10**) $\frac{5\pm3\sqrt{13}}{4}$

(**11**) $3\pm\sqrt{6}$

(**12**) $\frac{6\pm\sqrt{26}}{10}$

(**13**) $\frac{-1\pm\sqrt{29}}{4}$

(**14**) $\frac{1\pm\sqrt{46}}{5}$

(**15**) $\frac{1\pm\sqrt{61}}{10}$

(**16**) $\pm\sqrt{2}$

(**17**) $\frac{3\pm\sqrt{29}}{10}$

(**18**) $\frac{1\pm\sqrt{5}}{2}$

(**19**) $\frac{-3\pm\sqrt{73}}{4}$

(**20**) $\frac{-3\pm\sqrt{69}}{10}$

(**21**) $-2\pm\sqrt{6}$

(**22**) $1\pm\sqrt{10}$

(**23**) $\frac{-1\pm\sqrt{2}}{2}$

(**24**) $\frac{-1\pm\sqrt{13}}{4}$

(**25**) $\pm\sqrt{2}$

(**26**) $\frac{-5\pm\sqrt{5}}{2}$

(**27**) $\frac{-2\pm\sqrt{102}}{2}$

(**28**) $-2\pm\sqrt{13}$

(**29**) $\frac{2\pm\sqrt{19}}{5}$

(**30**) $\frac{3\pm\sqrt{41}}{2}$

(**31**) $\pm\sqrt{7}$

(**32**) $\frac{5\pm\sqrt{33}}{2}$

(**33**) $\frac{5\pm\sqrt{21}}{2}$

(**34**) $-2\pm\sqrt{7}$

(**35**) $\frac{5\pm\sqrt{41}}{4}$

(**36**) $\frac{-5\pm\sqrt{41}}{8}$

(**37**) $\frac{1\pm\sqrt{17}}{2}$

(**38**) $\frac{3\pm\sqrt{37}}{2}$

(**39**) $\frac{2\pm\sqrt{19}}{3}$

(**40**) $\frac{2\pm3\sqrt{6}}{5}$

(**41**) $\frac{-1\pm\sqrt{22}}{3}$

(**42**) $\pm\sqrt{6}$

(**43**) $\frac{1\pm\sqrt{17}}{2}$

(**44**) $\frac{-5\pm\sqrt{33}}{2}$

(**45**) $\frac{-3\pm3\sqrt{5}}{2}$

(**46**) $\pm\frac{\sqrt{10}}{2}$

(**47**) $2\pm\sqrt{5}$

(**48**) $\frac{-1\pm\sqrt{5}}{4}$

(**49**) $\frac{-1\pm\sqrt{17}}{4}$

(**50**) $\frac{5\pm\sqrt{37}}{6}$

(**51**) $\frac{-5\pm3\sqrt{5}}{10}$

(**52**) $\frac{-1\pm\sqrt{46}}{5}$

(**53**) $\frac{-1\pm\sqrt{5}}{2}$

(**54**) $\pm\frac{\sqrt{10}}{2}$

(**55**) $\frac{5\pm\sqrt{13}}{2}$

(**56**) $\frac{3\pm\sqrt{65}}{4}$

Answers to Exercise 7.7

(**1**) two	(**9**) none	(**17**) none
(**2**) two	(**10**) one	(**18**) none
(**3**) two	(**11**) one	(**19**) factors
(**4**) one	(**12**) two	(**20**) does not factor
(**5**) two	(**13**) two irrational	(**21**) factors
(**6**) one	(**14**) two rational	(**22**) does not factor
(**7**) none	(**15**) none	(**23**) does not factor
(**8**) none	(**16**) two irrational	(**24**) factors

(**25**) The claim is always true. As long as a and c have opposite signs, the product $4ac$ is negative. This means $-4ac$ is positive and the discriminate b^2-4ac cannot be negative. Therefore, the solutions are real numbers. [Note, the solutions are real numbers regardless of the value of b. Also, if $4ac$ happens to equal b^2, there is one real number solution, multiplicity two.]

(**26**) Consider the two linear equations $Ax+B=0$ and $Cx+D=0$ where neither A nor C equal 0. Each has a solution in the real numbers, so the equation $(Ax+B)(Cx+D)=0$ has at least one solution in the real numbers. If the discriminant of $ax^2+bx+c=0$, $a\neq 0$, is negative, then $ax^2+bx+c=0$ has no solution in the real numbers. Since $(Ax+B)(Cx+D)=0$ has at least one solution and $ax^2+bx+c=0$ has no solution, the equations cannot be equivalent and $(Ax+B)(Cx+D)\neq ax^2+bx+c$. [Note: in a subsequent course you will learn that if a quadratic expression factors, it factors into a product of linear expressions. Once you know that, you can prove the stronger claim that if the discriminant of $ax^2+bx+c=0$ is negative, then ax^2+bx+c cannot be factored in the real numbers.]

Answers to Exercise 7.8

(**1**) $4,5$

(**2**) $-8,6$

(**3**) $7,8,9$

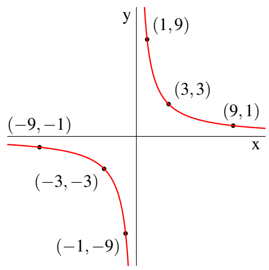
(**4**) 8 inches

(**5**) 2.5 feet

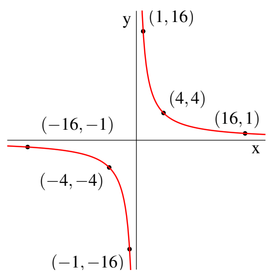
(**6**) The shop bought 20 bicycles.

Answers to Exercise 8.1

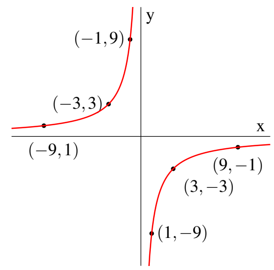
(**1**)



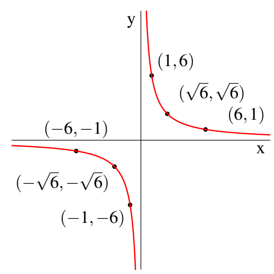
(**2**)



(**3**)



(**4**)



(**5**) Calculator: 2.236. Ours: 2.23. Difference: 0.006.
(**6**) Calculator: 4.243. Ours: 4.25. Difference: 0.007.
(**7**) Calculator: 4.472. Ours: 4.5. Difference: 0.028.
(**8**) Calculator: 8.246. Ours: 8.25. Difference: 0.004.
(**9**) Calculator: 8.485. Ours: 8.5. Difference: 0.015.
(**10**) Calculator: 10.440. Ours: 10.5. Difference: 0.06.

(**11**) $y = \frac{49}{x}$

(**12**) $y = \frac{13}{x}$

(**13**) $y = \frac{11}{x}$

(**14**) $y = \frac{8}{x}$

(**15**) $y = \frac{-5}{x}$

(**16**) $y = \frac{-2}{x}$

(**17**) $y = \frac{-36}{x}$

Answers to Exercise 8.2

(**1**) 3
(**2**) 6
(**3**) 15
(**4**) 36
(**5**) 98
(**6**) -1
(**7**) 10
(**8**) -1
(**9**) $2/9$
(**10**) 144
(**11**) $P(x) = \frac{15x}{1000}$, \$300
(**12**) $A(x) = \frac{x}{10}$, 100 gallons.

Answers to Exercise 8.3

(**1**) $\frac{1}{x}$ shifted 4 units right.
(**2**) $\frac{1}{x}$ shifted 6 units left.
(**3**) $\frac{2}{x}$ shifted 1 unit right.
(**4**) $\frac{3}{x}$ shifted 5 units left.
(**5**) $\frac{1/2}{x}$ shifted 4 units right.
(**6**) $\frac{1/5}{x}$ shifted 3 units left.
(**7**) $\frac{1}{x}$ shifted 2 units right.

Answers to Exercise 8.4

(**1**) Asymptotes: $x = 6, y = 0$. Domain: all real numbers except 6. Range: all real numbers except 0.

(**2**) Asymptotes: $x = -3, y = 0$. Domain: all real numbers except -3. Range: all real numbers except 0.

(**3**) Asymptotes: $x = 7, y = 1$. Domain: all real numbers except 7. Range: all real numbers except 1.

(**4**) Asymptotes: $x = 4, y = -3$. Domain: all real numbers except 4. Range: all real numbers except -3.

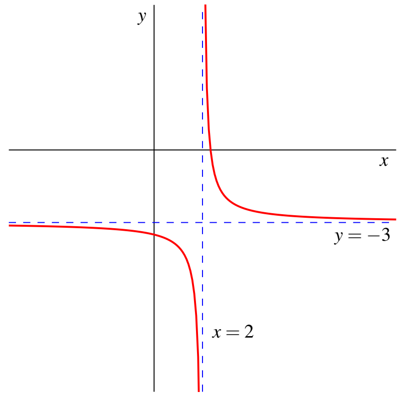
(**5**) Asymptotes: $x = -4, y = 0$. Domain: all real numbers except -4. Range: all real numbers except 0.

(**6**) Asymptotes: $x = -4, y = 0$. Domain: all real numbers except -4. Range: all real numbers except 0.

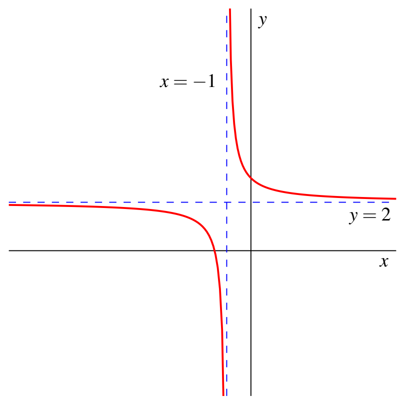
(**7**) Asymptotes: $x = 3, y = 4$. Domain: all real numbers except 3. Range: all real numbers except 4.

(**8**) Asymptotes: $x = -2, y = -6$. Domain: all real numbers except -2. Range: all real numbers except -6.

(**9**)



(**10**)



Answers to Exercise 9.1

(**1**) Let $y = x^2$. Suppose the portion of the graph from (x_1, y_1) to (x_2, y_2), $x_1 \neq x_2$, is straight. The function is defined for all real numbers, so there is a number x between x_1 and x_2, $x \neq x_1, x \neq x_2$ such that (x, y) is on $y = x^2$. Since we suppose the line is straight, the slopes $\frac{y_2 - y}{x_2 - x}$ and $\frac{y_1 - y}{x_1 - x}$ must be equal. So

$$\frac{y_2 - y}{x_2 - x} = \frac{y_1 - y}{x_1 - x}$$

$$\frac{x_2^2 - x^2}{x_2 - x} = \frac{x_1^2 - x^2}{x_1 - x}$$

$$x_2 + x = x_1 + x$$

$$x_2 = x_1.$$

But this contradicts that x_1 and x_2 are supposed to be different numbers. Therefore, no portion of $y = x^2$ is a straight line.

(**2**) Use Theorem 9.1 on page 203. $f(-x) = (-x)^4 = x^4 = f(x)$.

(**3**) $(-3, 9)$

(**4**) x tripled is $3x$. The $f(3x) = (3x)^2 = 9x^2 = 9f(x)$.

(**5**) x halved is $x/2$. The $f(x/2) = (x/2)^2 = \frac{x^2}{4} = \frac{1}{4}f(x)$.

(**6**) $(0, 0)$ and $(1, 1)$.

(**7**) For $0 < x < 1$

(**8**) For $0 < x < 5$

Answers to Exercise 9.2

(**1**) $f(x) = (x+6)^2 - 6$

(**2**) $f(x) = (x-5)^2 + 9$

(**3**) $f(x) = (x-7)^2 + 7$

(**4**) $f(x) = 3(x-2)^2$

(**5**) $f(x) = -(x-3)^2 + 5$

(**6**) $f(x) = -2(x+1)^2 + 1$

(**7**) $f(x) = 3(x+2)^2 - 7$

(**8**) $f(x) = 3(x-0)^2 - 8$

(**9**) $f(x) = -3(x+4)^2 - 10$

(**10**) $f(x) = (x+6)^2 - 1$

(**11**) concave down

(**12**) concave down

(**13**) concave up

(**14**) concave up

(**15**) $(3,-10)$, Concave up

(**16**) $(-4,-9)$, concave down

(**17**) $(-8,-2)$, concave up

(**18**) $(-2,-8)$, concave down

(**19**) $(-5,-1)$, concave down

(**20**) $(0,1)$, concave down

(**21**) $(-7,7)$, concave down

(**22**) $(4,-3)$, concave up

(**23**) $(7,-7)$, concave down

(**24**) $(4,0)$, concave up

(**25**) Min value $= 2$

(**26**) Min value $= 1$

(**27**) Min value $= 4$

(**28**) Min value $= -1$

(**29**) Min value $= 1$

(**30**) Max value $= -7$

(**31**) Max value $= 8$

(**32**) Min value $= -9$

(**33**) Min value $= -1$

(**34**) Max value $= -4$

(**35**) All numbers no greater than 3

(**36**) All numbers no less than -4

(**37**) All numbers no greater than -3

(**38**) All numbers no greater than 3

(**39**) All numbers no less than -6

(**40**) All numbers no greater than -7

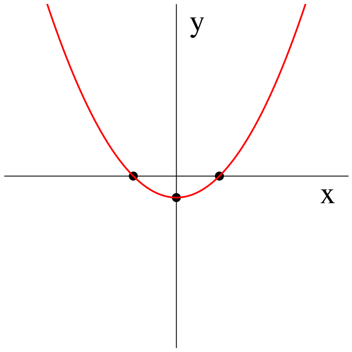
(**41**) All numbers no less than -10

(**42**) All numbers no less than -7

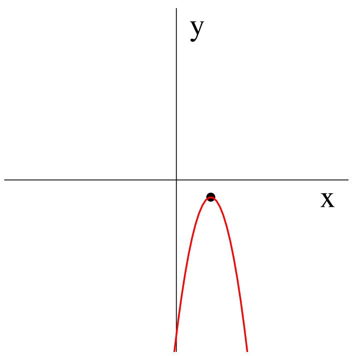
(**43**) All numbers no less than 6

(**44**) All numbers no less than -6

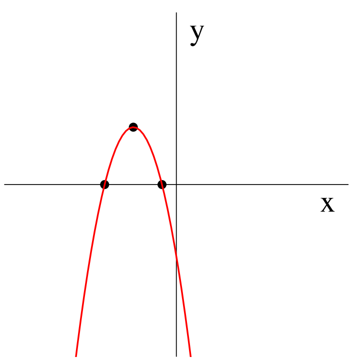
(**45**) Vertex: $(0,-1)$, x-int: 2 and -2.



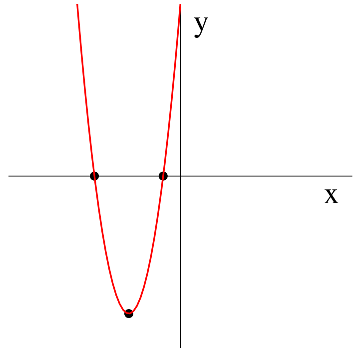
(**46**) V: $(2,-1)$, x-int: None.



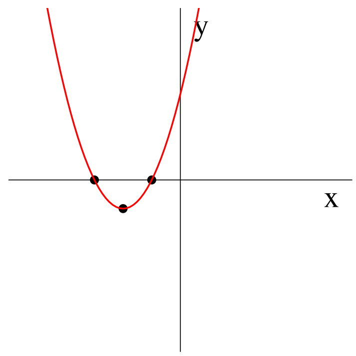
(**47**) V: $(-3,4)$, x-int: -5 and -1



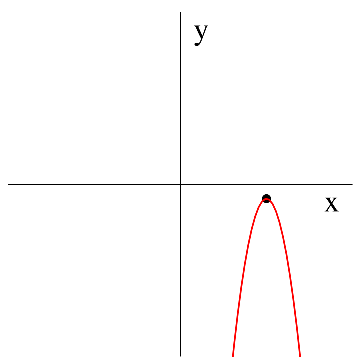
(**48**) V: $(-3,-8)$, x-int: -5 and -1



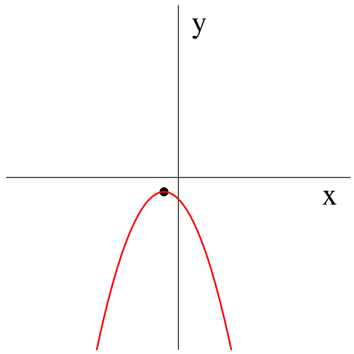
(**49**) V: $(-2,-1)$, x-int: -3 and -1



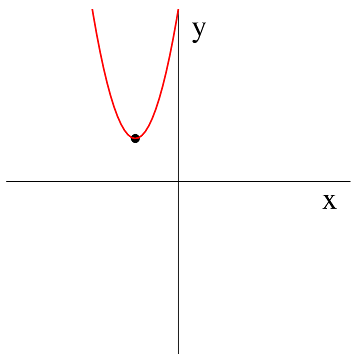
(**50**) V: $(6,-1)$, x-int: None



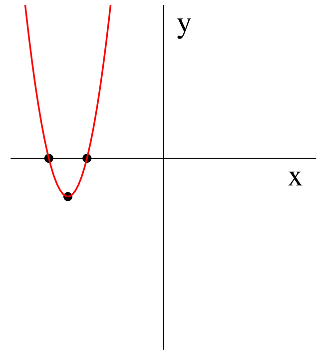
(**51**) V: $(-1,-1)$, x-int: None



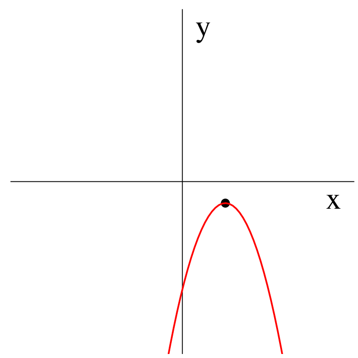
(**52**) V: $(-3,3)$, x-int: None



(**53**) V: $(-5,-2)$, x-int: -6 and -4



(**54**) V: $(2,-1)$, x-int: None



Answers to Exercise 9.3

(**1**) $x=-3, x=8$

(**2**) $x=-\sqrt{2},\ x=\sqrt{2}$
$x\approx -1.4,\ x\approx 1.4$

(**3**) $x=3-\sqrt{2},\ x=3+\sqrt{2}$
$x\approx 1.6,\ x\approx 4.4$

(**4**) $x=1-\sqrt{5},\ x=1+\sqrt{5}$
$x\approx -1.2,\ x\approx 3.2$

(**5**) $x=-2$

(**6**) None

(**7**) $x=2-\sqrt{6},\ x=2+\sqrt{6}$
$x\approx -0.5,\ x\approx 4.5$

(**8**) $x=-\frac{2}{3},\ x=-\frac{1}{2}$
$x\approx -0.7,\ x=-0.5$

(**9**) $x=-4,\ x=-\frac{4}{3}$
$x\approx -1.3,\ x=-4$

(**10**) None

Answers to Exercise 9.4

(**1**) Suppose $0 < x_1 < x_2$. Consider the difference $f(x_2) - f(x_1)$.

$$f(x_2) - f(x_1) = x_2^2 - x_1^2$$

$$= (x_2 - x_1)(x_2 + x_1).$$

Since each factor is positive,

$$f(x_2) - f(x_1) > 0.$$

Therefore by Definition 9.2 on page 218, f is an increasing function on $x > 0$.
(**2**) Suppose $5 < x_1 < x_2$. Consider the difference $f(x_2) - f(x_1)$.

$$f(x_2) - f(x_1) = x_2^2 - 10x_2 + 16 - (x_1^2 - 10x_1 + 16)$$

$$= x_2^2 - 10x_2 + 16 - x_1^2 + 10x_1 - 16)$$

$$= x_2^2 - x_1^2 - 10(x_2 - x_1)$$

$$= (x_2 - x_1)(x_2 + x_1 - 10).$$

$x_2 - x_1 > 0$, because $x_2 > x_1$. Since x_1 and x_2 each greater than 5, $x_2 + x_1 - 10 > 0$. Thus,

$$f(x_2) - f(x_1) > 0.$$

Therefore by Definition 9.2 on page 218, f is an increasing function on $x > 5$.

(**3**) f is increasing on $x < -2$ and $x > 1$. f decreasing on $-2 < x < 1$.

(**4**) f is increasing on $x < -1$ and $x > 3$. f decreasing on $-1 < x < 3$.

(**5**) Let $P(x_1, y_1)$ be any fixed point on the graph of $y = mx^2 + b$. Let $Q(x, y)$ be any other variable point on the graph. Then

$$y_1 = mx_1^2 + b.$$

$$y = mx^2 + b.$$

$$\text{Slope}_{PQ} = \frac{(mx_1^2 + b) - (mx^2 + b)}{x_2 - x}$$

$$= \frac{mx_1^2 - mx^2}{x_1 - x}$$

$$= m\left(\frac{x_1^2 - x^2}{x_1 - x}\right)$$

$$= m(x_1 + x).$$

But, as x takes various values, so does Slope_{PQ}. That is no one's idea of a constant slope.

(**6**) Select three numbers x_1, x_2, x_3 in the domain of $f(x) = mx^2 + bx + c, m \neq 0$, such that $x_1 < x_2 < x_3$. The corresponding values of the function are

$$x_1 \longrightarrow mx_1^2 + bx_1 + c$$

$$x_2 \longrightarrow mx_2^2 + bx_2 + c$$

$$x_3 \longrightarrow mx_3^2 + bx_3 + c.$$

Compute slopes using two pairs of points.

$$\text{slope}_1 = \frac{mx_2^2 + bx_2 + c - (mx_1^2 + bx_1 + c)}{x_2 - x_1}$$

$$= \frac{mx_2^2 + bx_2 + c - mx_1^2 - bx_1 - c}{x_2 - x_1}$$

$$= \frac{mx_2^2 - mx_1^2 + bx_2 - bx_1}{x_2 - x_1}$$

$$= \frac{m(x_2^2 - x_1^2) + b(x_2 - x_1)}{x_2 - x_1}$$

$$= \frac{m(x_2 - x_1)(x_2 + x_1) + b(x_2 - x_1)}{x_2 - x_1}$$

$$= m(x_2 + x_1) + b.$$

Computing slope_2 similarly using x_1 and x_3 will result in

$$\text{slope}_2 = m(x_3 + x_1) + b.$$

If the portion of the graph is straight, then $\text{slope}_2 = \text{slope}_2$ which means

$$m(x_3 + x_1) + b = m(x_2 + x_1) + b$$

$$x_3 + x_1 = x_2 + x_1$$

$$x_3 = x_2.$$

This contradicts the supposition that $x_1 < x_2 < x_3$. So the portion of the graph cannot be straight.

Answers to Exercise 10.1

(**1**) Suppose $y = 2x + 3$ pairs $(x_1, y_1), (x_2, y_2)$. Now suppose $x_1 = x_2$. Then

$$y_1 = 2x_1 + 3 \iff x_1 = \frac{y_1 - 3}{2}$$

and

$$y_2 = 2x_2 + 3 \iff x_2 = \frac{y_2 - 3}{2}.$$

Since we suppose $x_1 = x_2$

$$, \frac{y_1 - 3}{2} = \frac{y_2 - 3}{2}$$

$$y_1 - 3 = y_2 - 3$$

$$y_1 = y_2.$$

(**2**) Let $y = \frac{1}{x}, x > 0$. Suppose (x_1, y_1) and (x_2, y_2) satisfy $y = \frac{1}{x}$. Further suppose that $x_1 = x_2$. Then

$$y_1 = \frac{1}{x_1} \iff x_1 = \frac{1}{y_1}$$

and

$$y_2 = \frac{1}{x_2} \iff x_2 = \frac{1}{y_2}.$$

Since $x_1 = x_2$,

$$\frac{1}{y_1} = \frac{1}{y_2}$$

$$y_1 = y_2.$$

(**3**) Let $f(x) = a_1x^2 + b_1x + c_1$ and $g(x) = a_2x^2 + b_2x + c_2$. Then $h(x) = f(x) + g(x) = a_1x^2 + b_1x + c_1 + a_2x^2 + b_2x + c_2$. Combine like terms, $h(x) = (a_1 + a_2)x^2 + (b_1 + b_2)x + c_1 + c_2$. Combine constants $h(x) = ax^2 + bx + c$.

(**4**) When $k = 0$.
(**5**) Yes. In fact it is called a "constant function".
(**6**) $2^2 = 4 = (-2)^2$, but $2 \neq -2$.

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