EXERCISES 10.11

16-02-04-12S

1–10 ▪ Use the binomial series to expand the given function as a power series. State the radius of convergence.

1. $\sqrt{1+x}$
2. $\dfrac{1}{(1+x)^3}$
3. $\dfrac{1}{(1+2x)^4}$
4. $\sqrt[3]{1+x^2}$
5. $\dfrac{x}{\sqrt{1-x}}$
6. $\dfrac{1}{\sqrt{2+x}}$
7. $\sqrt[4]{1-x^4}$
8. $\dfrac{x^2}{\sqrt{1-x^3}}$
9. $\left(\dfrac{x}{1-x}\right)^5$
10. $\sqrt[5]{x-1}$

11–12 ▪ Use the binomial series to expand the given function as a Maclaurin series and to find the first three Taylor polynomials T_1, T_2, and T_3. Graph the function and these Taylor polynomials in the interval of convergence.

11. $\dfrac{1}{\sqrt[3]{8+x}}$
12. $(4+x)^{3/2}$

13. (a) Use the binomial series to expand $1/\sqrt{1-x^2}$.
    (b) Use part (a) to find the Maclaurin series for $\sin^{-1}x$.

14. (a) Use the binomial series to expand $1/\sqrt{1+x^2}$.
    (b) Use part (a) to find the Maclaurin series for $\sinh^{-1}x$.

15. (a) Expand $1/\sqrt{1+x}$ as a power series.
    (b) Use part (a) to estimate $1/\sqrt{1.1}$ correct to three decimal places.

16. (a) Expand $\sqrt[3]{8+x}$ as a power series.
    (b) Use part (a) to estimate $\sqrt[3]{8.2}$ correct to four decimal places.

17. (a) Expand $f(x) = x/(1-x)^2$ as a power series.
    (b) Use part (a) to find the sum of the series

$$\sum_{n=1}^{\infty} \frac{n}{2^n}$$

18. (a) Expand $f(x) = (x+x^2)/(1-x)^3$ as a power series.
    (b) Use part (a) to find the sum of the series

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

19. (a) Use the binomial series to find the Maclaurin series of $f(x) = \sqrt{1+x^2}$.
    (b) Use part (a) to evaluate $f^{(10)}(0)$.

20. (a) Use the binomial series to find the Maclaurin series of $f(x) = 1/\sqrt{1+x^3}$.
    (b) Use part (a) to evaluate $f^{(9)}(0)$.

21. Use the following steps to prove (2).
    (a) Let $g(x) = \sum_{n=0}^{\infty} \binom{k}{n} x^n$. Differentiate this series to show that

$$g'(x) = \frac{kg(x)}{1+x} \qquad -1 < x < 1$$

    (b) Let $h(x) = (1+x)^{-k} g(x)$ and show that $h'(x) = 0$.
    (c) Deduce that $g(x) = (1+x)^k$.

22. The period of a pendulum with length L that makes a maximum angle θ_0 with the vertical is

$$T = 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{dx}{\sqrt{1-k^2\sin^2 x}}$$

where $k = \sin(\frac{1}{2}\theta_0)$ and g is the acceleration due to gravity. (In Exercise 35 in Section 7.8 we approximated this integral using Simpson's Rule.)

(a) Expand the integrand as a binomial series and use the result of Exercise 40 in Section 7.1 to show that

$$T = 2\pi\sqrt{\frac{L}{g}}\left[1 + \frac{1^2}{2^2}k^2 + \frac{1^2 3^2}{2^2 4^2}k^4 + \frac{1^2 3^2 5^2}{2^2 4^2 6^2}k^6 + \cdots\right]$$

If θ_0 is not too large, the approximation $T \approx 2\pi\sqrt{L/g}$, obtained by using only the first term in the series, is often used. A better approximation is obtained by using two terms:

$$T \approx 2\pi\sqrt{\frac{L}{g}}\left(1 + \tfrac{1}{4}k^2\right)$$

(b) Notice that all the terms in the series after the first one have coefficients that are at most $\frac{1}{4}$. Use this fact to compare this series with a geometric series and show that

$$2\pi\sqrt{\frac{L}{g}}\left(1 + \tfrac{1}{4}k^2\right) \le T \le 2\pi\sqrt{\frac{L}{g}}\,\frac{4-3k^2}{4-4k^2}$$

(c) Use the inequalities in part (b) to estimate the period of a pendulum with $L = 1$ meter and $\theta_0 = 10°$. How does it compare with the estimate $T \approx 2\pi\sqrt{L/g}$? What if $\theta_0 = 42°$?

37. 0.310 **39.** 0.09998750 **41.** $1 - \frac{3}{2}x^2 + \frac{25}{4}x^4$
43. $-x + \frac{1}{2}x^2 - \frac{1}{3}x^3$ **45.** e^{-x^4} **47.** $1/\sqrt{2}$ **49.** $e^x - 1$
53. 1/120

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1. $1 + \frac{x}{2} + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)}{2^n n!} x^n$, $R = 1$

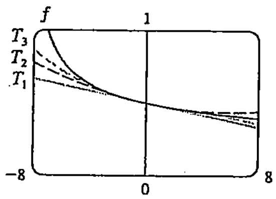
3. $\sum_{n=0}^{\infty} (-1)^n \frac{(n+1)(n+2)(n+3)2^n}{6} x^n$, $R = \frac{1}{2}$

5. $x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2^n n!} x^{n+1}$, $R = 1$

7. $1 - \frac{x^4}{4} - \sum_{n=2}^{\infty} \frac{3 \cdot 7 \cdot 11 \cdot \cdots \cdot (4n-5)}{4^n n!} x^{4n}$, $R = 1$

9. $\sum_{n=0}^{\infty} \frac{(n+4)!}{4! \cdot n!} x^{n+5}$, $R = 1$

11. $\frac{1}{2} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 4 \cdot 7 \cdot \cdots \cdot (3n-2)}{24^n n!} x^n$, $R = 8$



13. (a) $1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2^n n!} x^{2n}$

(b) $x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2^n n!} \frac{x^{2n+1}}{2n+1}$

15. (a) $1 + \sum_{n=1}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2^n n!} x^n$ (b) 0.953

17. (a) $\sum_{n=1}^{\infty} nx^n$ (b) 2

19. (a) $1 + \frac{x^2}{2} + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)}{2^n n!} x^{2n}$
(b) 99,225

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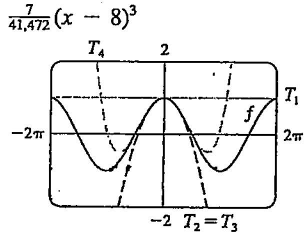
1. $\frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{6}\right)^3$

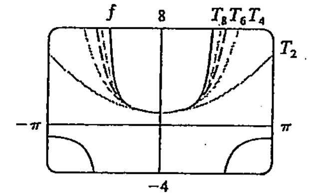
3. $x + \frac{1}{3}x^3$ **5.** $x + x^2 + \frac{1}{3}x^3$

7. $\frac{1}{2} - \frac{1}{48}(x-8) + \frac{1}{576}(x-8)^2 - \frac{7}{41{,}472}(x-8)^3$

9. $T_1(x) = 1$, $T_2(x) = 1 - \frac{1}{2}x^2$,
$T_3(x) = 1 - \frac{1}{2}x^2$,
$T_4(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$



11. $T_8(x) = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \frac{277}{8064}x^8$



13. (a) $1 + \frac{1}{2}x$ (b) 0.00125

15. (a) $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) - \frac{1}{2\sqrt{2}}\left(x - \frac{\pi}{4}\right)^2 - \frac{1}{6\sqrt{2}}\left(x - \frac{\pi}{4}\right)^3 + \frac{1}{24\sqrt{2}}\left(x - \frac{\pi}{4}\right)^4 + \frac{1}{120\sqrt{2}}\left(x - \frac{\pi}{4}\right)^5$
(b) 0.00033 **17.** (a) $x + \frac{1}{3}x^3$ (b) 0.06

19. (a) $1 + x^2$ (b) 0.00006

21. (a) $8 + \frac{3}{8}(x-16) - \frac{3}{1024}(x-16)^2 + \frac{5}{65{,}536}(x-16)^3$
(b) 0.0000034 **23.** 0.57358 **25.** 3

27. $-1.037 < x < 1.037$ **29.** 21 m, no

Review Exercises for Chapter 10 ■ page 655

1. False **3.** False **5.** False **7.** False **9.** False
11. True **13.** True **15.** False **17.** True
19. C, $\frac{1}{2}$ **21.** D **23.** D **25.** C, e^{12} **27.** 2
29. D **31.** C **33.** C **35.** C **37.** C **39.** D
41. CC **43.** AC **45.** 8 **47.** $\pi/4$ **49.** $\frac{4111}{3330}$
51. 0.9721 **53.** 0.18976224, error $< 6.4 \times 10^{-7}$
57. 3, $[-3, 3]$ **59.** 0.5, $[2.5, 3.5)$

61. $\frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{2}\frac{1}{2!}\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{2}\frac{1}{3!}\left(x - \frac{\pi}{6}\right)^3 + \cdots$
$= \frac{1}{2}\sum_{n=0}^{\infty} (-1)^n \left[\frac{1}{(2n)!}\left(x - \frac{\pi}{6}\right)^{2n} + \frac{\sqrt{3}}{(2n+1)!}\left(x - \frac{\pi}{6}\right)^{2n+1}\right]$

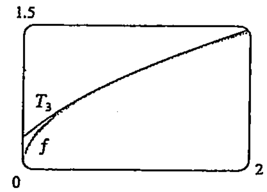
63. $\sum_{n=0}^{\infty} (-1)^n x^{n+2}$, 1 **65.** $-\sum_{n=1}^{\infty} \frac{x^n}{n}$, 1

67. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{8n+4}}{(2n+1)!}$, ∞

69. $\frac{1}{2} + \sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n-3)}{n!\, 2^{6n+1}} x^n$, 16

71. $\ln|x| + C + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}$

73. (a) $1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3$
(b)



(c) 0.000006 **75.** 1