

**10.6**

**Graphing in Polar Coordinates**

This section describes techniques for graphing equations in polar coordinates.

**Symmetry**

Figure 10.43 illustrates the standard polar coordinate tests for symmetry.

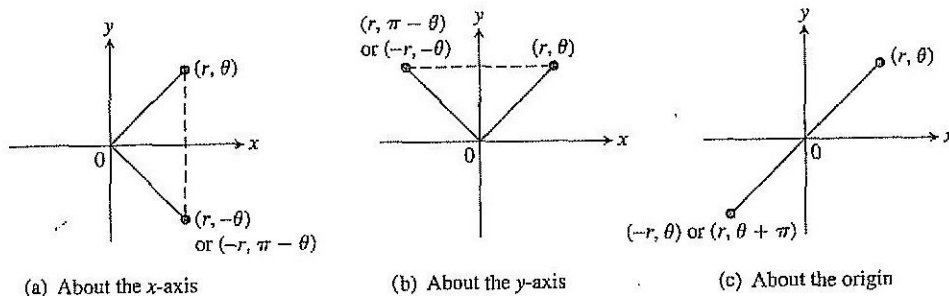


FIGURE 10.43 Three tests for symmetry in polar coordinates.

**Symmetry Tests for Polar Graphs**

1. *Symmetry about the x-axis:* If the point  $(r, \theta)$  lies on the graph, the point  $(r, -\theta)$  or  $(-r, \pi - \theta)$  lies on the graph (Figure 10.43a).
2. *Symmetry about the y-axis:* If the point  $(r, \theta)$  lies on the graph, the point  $(r, \pi - \theta)$  or  $(-r, -\theta)$  lies on the graph (Figure 10.43b).
3. *Symmetry about the origin:* If the point  $(r, \theta)$  lies on the graph, the point  $(-r, \theta)$  or  $(r, \theta + \pi)$  lies on the graph (Figure 10.43c).

**Slope**

The slope of a polar curve  $r = f(\theta)$  is given by  $dy/dx$ , not by  $r' = df/d\theta$ . To see why, think of the graph of  $f$  as the graph of the parametric equations

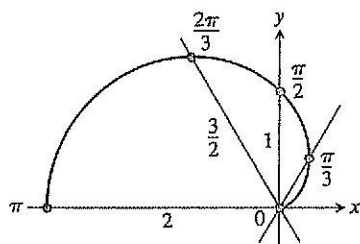
$$x = r \cos \theta = f(\theta) \cos \theta, \quad y = r \sin \theta = f(\theta) \sin \theta.$$

If  $f$  is a differentiable function of  $\theta$ , then so are  $x$  and  $y$  and, when  $dx/d\theta \neq 0$ , we can calculate  $dy/dx$  from the parametric formula

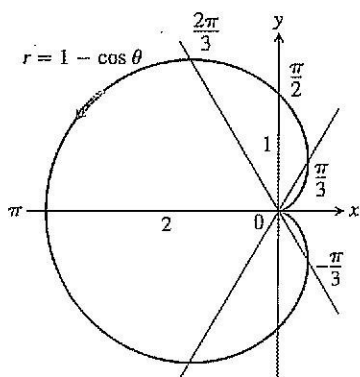
$$\begin{aligned} \frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} && \text{Section 3.5, Equation (2) with } t = \theta \\ &= \frac{\frac{d}{d\theta}(f(\theta) \cdot \sin \theta)}{\frac{d}{d\theta}(f(\theta) \cdot \cos \theta)} \\ &= \frac{\frac{df}{d\theta} \sin \theta + f(\theta) \cos \theta}{\frac{df}{d\theta} \cos \theta - f(\theta) \sin \theta} && \text{Product Rule for derivatives} \end{aligned}$$

| $\theta$         | $r = 1 - \cos \theta$ |
|------------------|-----------------------|
| 0                | 0                     |
| $\frac{\pi}{3}$  | $\frac{1}{2}$         |
| $\frac{\pi}{2}$  | 1                     |
| $\frac{2\pi}{3}$ | $\frac{3}{2}$         |
| $\pi$            | 2                     |

(a)



(b)



(c)

FIGURE 10.44 The steps in graphing the cardioid  $r = 1 - \cos \theta$  (Example 1). The arrow shows the direction of increasing  $\theta$ .

**Slope of the Curve  $r = f(\theta)$**

$$\left. \frac{dy}{dx} \right|_{(r, \theta)} = \frac{f'(\theta) \sin \theta + f(\theta) \cos \theta}{f'(\theta) \cos \theta - f(\theta) \sin \theta}$$

provided  $dx/d\theta \neq 0$  at  $(r, \theta)$ .

If the curve  $r = f(\theta)$  passes through the origin at  $\theta = \theta_0$ , then  $f(\theta_0) = 0$ , and the slope equation gives

$$\left. \frac{dy}{dx} \right|_{(0, \theta_0)} = \frac{f'(\theta_0) \sin \theta_0}{f'(\theta_0) \cos \theta_0} = \tan \theta_0.$$

If the graph of  $r = f(\theta)$  passes through the origin at the value  $\theta = \theta_0$ , the slope of the curve there is  $\tan \theta_0$ . The reason we say “slope at  $(0, \theta_0)$ ” and not just “slope at the origin” is that a polar curve may pass through the origin (or any point) more than once, with different slopes at different  $\theta$ -values. This is not the case in our first example, however.

**EXAMPLE 1 A Cardioid**

Graph the curve  $r = 1 - \cos \theta$ .

**Solution** The curve is symmetric about the  $x$ -axis because

$$\begin{aligned} (r, \theta) \text{ on the graph} &\Rightarrow r = 1 - \cos \theta \\ &\Rightarrow r = 1 - \cos(-\theta) && \cos \theta = \cos(-\theta) \\ &\Rightarrow (r, -\theta) \text{ on the graph.} \end{aligned}$$

As  $\theta$  increases from 0 to  $\pi$ ,  $\cos \theta$  decreases from 1 to  $-1$ , and  $r = 1 - \cos \theta$  increases from a minimum value of 0 to a maximum value of 2. As  $\theta$  continues on from  $\pi$  to  $2\pi$ ,  $\cos \theta$  increases from  $-1$  back to 1 and  $r$  decreases from 2 back to 0. The curve starts to repeat when  $\theta = 2\pi$  because the cosine has period  $2\pi$ .

The curve leaves the origin with slope  $\tan(0) = 0$  and returns to the origin with slope  $\tan(2\pi) = 0$ .

We make a table of values from  $\theta = 0$  to  $\theta = \pi$ , plot the points, draw a smooth curve through them with a horizontal tangent at the origin, and reflect the curve across the  $x$ -axis to complete the graph (Figure 10.44). The curve is called a *cardioid* because of its heart shape. Cardioid shapes appear in the cams that direct the even layering of thread on bobbins and reels, and in the signal-strength pattern of certain radio antennas. □

**EXAMPLE 2** Graph the Curve  $r^2 = 4 \cos \theta$ .

**Solution** The equation  $r^2 = 4 \cos \theta$  requires  $\cos \theta \geq 0$ , so we get the entire graph by running  $\theta$  from  $-\pi/2$  to  $\pi/2$ . The curve is symmetric about the  $x$ -axis because

$$\begin{aligned} (r, \theta) \text{ on the graph} &\Rightarrow r^2 = 4 \cos \theta \\ &\Rightarrow r^2 = 4 \cos(-\theta) && \cos \theta = \cos(-\theta) \\ &\Rightarrow (r, -\theta) \text{ on the graph.} \end{aligned}$$

The curve is also symmetric about the origin because

$$\begin{aligned} (r, \theta) \text{ on the graph} &\Rightarrow r^2 = 4 \cos \theta \\ &\Rightarrow (-r)^2 = 4 \cos \theta \\ &\Rightarrow (-r, \theta) \text{ on the graph.} \end{aligned}$$

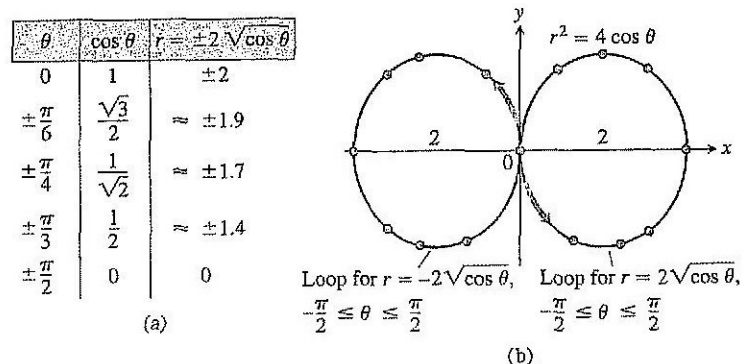
Together, these two symmetries imply symmetry about the  $y$ -axis.

The curve passes through the origin when  $\theta = -\pi/2$  and  $\theta = \pi/2$ . It has a vertical tangent both times because  $\tan \theta$  is infinite.

For each value of  $\theta$  in the interval between  $-\pi/2$  and  $\pi/2$ , the formula  $r^2 = 4 \cos \theta$  gives two values of  $r$ :

$$r = \pm 2\sqrt{\cos \theta}.$$

We make a short table of values, plot the corresponding points, and use information about symmetry and tangents to guide us in connecting the points with a smooth curve (Figure 10.45). ▣



**FIGURE 10.45** The graph of  $r^2 = 4 \cos \theta$ . The arrows show the direction of increasing  $\theta$ . The values of  $r$  in the table are rounded (Example 2).

### A Technique for Graphing

One way to graph a polar equation  $r = f(\theta)$  is to make a table of  $(r, \theta)$ -values, plot the corresponding points, and connect them in order of increasing  $\theta$ . This can work well if enough points have been plotted to reveal all the loops and dimples in the graph. Another method of graphing that is usually quicker and more reliable is to

1. first graph  $r = f(\theta)$  in the Cartesian  $r\theta$ -plane,
2. then use the Cartesian graph as a “table” and guide to sketch the polar coordinate graph.

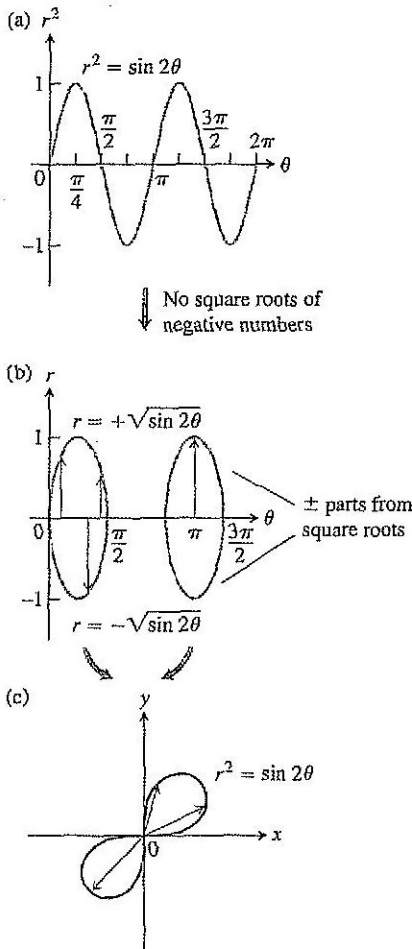
This method is better than simple point plotting because the first Cartesian graph, even when hastily drawn, shows at a glance where  $r$  is positive, negative, and nonexistent, as well as where  $r$  is increasing and decreasing. Here's an example.

**EXAMPLE 3** A Lemniscate

Graph the curve

$$r^2 = \sin 2\theta.$$

**Solution** Here we begin by plotting  $r^2$  (not  $r$ ) as a function of  $\theta$  in the Cartesian  $r^2\theta$ -plane. See Figure 10.46a. We pass from there to the graph of  $r = \pm\sqrt{\sin 2\theta}$  in the  $r\theta$ -plane (Figure 10.46b), and then draw the polar graph (Figure 10.46c). The graph in Figure 10.46b “covers” the final polar graph in Figure 10.46c twice. We could have managed with either loop alone, with the two upper halves, or with the two lower halves. The double covering does no harm, however, and we actually learn a little more about the behavior of the function this way. ▮



**FIGURE 10.46** To plot  $r = f(\theta)$  in the Cartesian  $r\theta$ -plane in (b), we first plot  $r^2 = \sin 2\theta$  in the  $r^2\theta$ -plane in (a) and then ignore the values of  $\theta$  for which  $\sin 2\theta$  is negative. The radii from the sketch in (b) cover the polar graph of the lemniscate in (c) twice (Example 3).

**Finding Points Where Polar Graphs Intersect**

The fact that we can represent a point in different ways in polar coordinates makes extra care necessary in deciding when a point lies on the graph of a polar equation and in determining the points in which polar graphs intersect. The problem is that a point of intersection may satisfy the equation of one curve with polar coordinates that are different from the ones with which it satisfies the equation of another curve. Thus, solving the equations of two curves simultaneously may not identify all their points of intersection. One sure way to identify all the points of intersection is to graph the equations.

**EXAMPLE 4** Deceptive Polar Coordinates

Show that the point  $(2, \pi/2)$  lies on the curve  $r = 2 \cos 2\theta$ .

**Solution** It may seem at first that the point  $(2, \pi/2)$  does not lie on the curve because substituting the given coordinates into the equation gives

$$2 = 2 \cos 2\left(\frac{\pi}{2}\right) = 2 \cos \pi = -2,$$

which is not a true equality. The magnitude is right, but the sign is wrong. This suggests looking for a pair of coordinates for the same given point in which  $r$  is negative, for example,  $(-2, -(\pi/2))$ . If we try these in the equation  $r = 2 \cos 2\theta$ , we find

$$-2 = 2 \cos 2\left(-\frac{\pi}{2}\right) = 2(-1) = -2,$$

and the equation is satisfied. The point  $(2, \pi/2)$  does lie on the curve. ▮

**EXAMPLE 5** Elusive Intersection Points

Find the points of intersection of the curves

$$r^2 = 4 \cos \theta \quad \text{and} \quad r = 1 - \cos \theta.$$

## HISTORICAL BIOGRAPHY

Johannes Kepler  
(1571–1630)

**Solution** In Cartesian coordinates, we can always find the points where two curves cross by solving their equations simultaneously. In polar coordinates, the story is different. Simultaneous solution may reveal some intersection points without revealing others. In this example, simultaneous solution reveals only two of the four intersection points. The others are found by graphing. (Also, see Exercise 49.)

If we substitute  $\cos \theta = r^2/4$  in the equation  $r = 1 - \cos \theta$ , we get

$$r = 1 - \cos \theta = 1 - \frac{r^2}{4}$$

$$4r = 4 - r^2$$

$$r^2 + 4r - 4 = 0$$

$$r = -2 \pm 2\sqrt{2}.$$

Quadratic formula

The value  $r = -2 - 2\sqrt{2}$  has too large an absolute value to belong to either curve. The values of  $\theta$  corresponding to  $r = -2 + 2\sqrt{2}$  are

$$\theta = \cos^{-1}(1 - r) \quad \text{From } r = 1 - \cos \theta$$

$$= \cos^{-1}(1 - (2\sqrt{2} - 2)) \quad \text{Set } r = 2\sqrt{2} - 2.$$

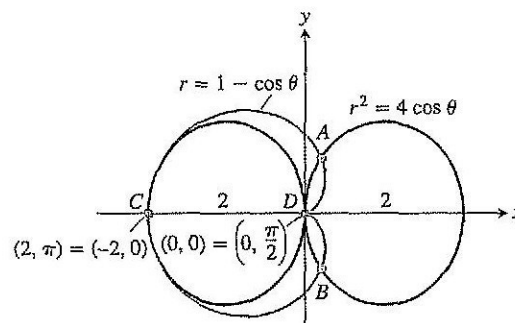
$$= \cos^{-1}(3 - 2\sqrt{2})$$

$$= \pm 80^\circ.$$

Rounded to the nearest degree

We have thus identified two intersection points:  $(r, \theta) = (2\sqrt{2} - 2, \pm 80^\circ)$ .

If we graph the equations  $r^2 = 4 \cos \theta$  and  $r = 1 - \cos \theta$  together (Figure 10.47), as we can now do by combining the graphs in Figures 10.44 and 10.45, we see that the curves also intersect at the point  $(2, \pi)$  and the origin. Why weren't the  $r$ -values of these points revealed by the simultaneous solution? The answer is that the points  $(0, 0)$  and  $(2, \pi)$  are not on the curves "simultaneously." They are not reached at the same value of  $\theta$ . On the curve  $r = 1 - \cos \theta$ , the point  $(2, \pi)$  is reached when  $\theta = \pi$ . On the curve  $r^2 = 4 \cos \theta$ , it is reached when  $\theta = 0$ , where it is identified not by the coordinates  $(2, \pi)$ , which do not satisfy the equation, but by the coordinates  $(-2, 0)$ , which do. Similarly, the cardioid reaches the origin when  $\theta = 0$ , but the curve  $r^2 = 4 \cos \theta$  reaches the origin when  $\theta = \pi/2$ .  $\square$



**FIGURE 10.47** The four points of intersection of the curves  $r = 1 - \cos \theta$  and  $r^2 = 4 \cos \theta$  (Example 5). Only  $A$  and  $B$  were found by simultaneous solution. The other two were disclosed by graphing.

**USING TECHNOLOGY** Graphing Polar Curves Parametrically

For complicated polar curves we may need to use a graphing calculator or computer to graph the curve. If the device does not plot polar graphs directly, we can convert  $r = f(\theta)$  into parametric form using the equations

$$x = r \cos \theta = f(\theta) \cos \theta, \quad y = r \sin \theta = f(\theta) \sin \theta.$$

Then we use the device to draw a parametrized curve in the Cartesian  $xy$ -plane. It may be required to use the parameter  $t$  rather than  $\theta$  for the graphing device.

**EXERCISES 10.6****Symmetries and Polar Graphs**

Identify the symmetries of the curves in Exercises 1–12. Then sketch the curves.

- |                          |                            |
|--------------------------|----------------------------|
| 1. $r = 1 + \cos \theta$ | 2. $r = 2 - 2 \cos \theta$ |
| 3. $r = 1 - \sin \theta$ | 4. $r = 1 + \sin \theta$   |
| 5. $r = 2 + \sin \theta$ | 6. $r = 1 + 2 \sin \theta$ |
| 7. $r = \sin(\theta/2)$  | 8. $r = \cos(\theta/2)$    |
| 9. $r^2 = \cos \theta$   | 10. $r^2 = \sin \theta$    |
| 11. $r^2 = -\sin \theta$ | 12. $r^2 = -\cos \theta$   |

Graph the lemniscates in Exercises 13–16. What symmetries do these curves have?

- |                            |                            |
|----------------------------|----------------------------|
| 13. $r^2 = 4 \cos 2\theta$ | 14. $r^2 = 4 \sin 2\theta$ |
| 15. $r^2 = -\sin 2\theta$  | 16. $r^2 = -\cos 2\theta$  |

**Slopes of Polar Curves**

Find the slopes of the curves in Exercises 17–20 at the given points. Sketch the curves along with their tangents at these points.

17. **Cardioid**  $r = -1 + \cos \theta$ ;  $\theta = \pm\pi/2$   
 18. **Cardioid**  $r = -1 + \sin \theta$ ;  $\theta = 0, \pi$   
 19. **Four-leaved rose**  $r = \sin 2\theta$ ;  $\theta = \pm\pi/4, \pm3\pi/4$   
 20. **Four-leaved rose**  $r = \cos 2\theta$ ;  $\theta = 0, \pm\pi/2, \pi$

**Limaçons**

Graph the limaçons in Exercises 21–24. Limaçon (“lee-ma-sahn”) is Old French for “snail.” You will understand the name when you graph the limaçons in Exercise 21. Equations for limaçons have the form  $r = a \pm b \cos \theta$  or  $r = a \pm b \sin \theta$ . There are four basic shapes.

**21. Limaçons with an inner loop**

|                                    |                                    |
|------------------------------------|------------------------------------|
| a. $r = \frac{1}{2} + \cos \theta$ | b. $r = \frac{1}{2} + \sin \theta$ |
|------------------------------------|------------------------------------|

**22. Cardioids**

|                          |                           |
|--------------------------|---------------------------|
| a. $r = 1 - \cos \theta$ | b. $r = -1 + \sin \theta$ |
|--------------------------|---------------------------|

**23. Dimpled limaçons**

|                                    |                                    |
|------------------------------------|------------------------------------|
| a. $r = \frac{3}{2} + \cos \theta$ | b. $r = \frac{3}{2} - \sin \theta$ |
|------------------------------------|------------------------------------|

**24. Oval limaçons**

|                          |                           |
|--------------------------|---------------------------|
| a. $r = 2 + \cos \theta$ | b. $r = -2 + \sin \theta$ |
|--------------------------|---------------------------|

**Graphing Polar Inequalities**

25. Sketch the region defined by the inequalities  $-1 \leq r \leq 2$  and  $-\pi/2 \leq \theta \leq \pi/2$ .  
 26. Sketch the region defined by the inequalities  $0 \leq r \leq 2 \sec \theta$  and  $-\pi/4 \leq \theta \leq \pi/4$ .

In Exercises 27 and 28, sketch the region defined by the inequality.


|                                       |                                   |
|---------------------------------------|-----------------------------------|
| 27. $0 \leq r \leq 2 - 2 \cos \theta$ | 28. $0 \leq r^2 \leq \cos \theta$ |
|---------------------------------------|-----------------------------------|

**Intersections**

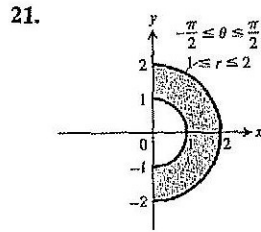
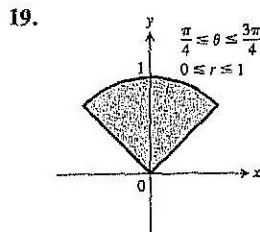
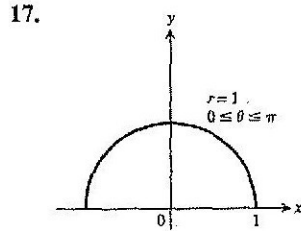
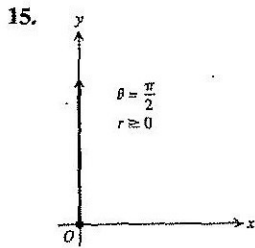
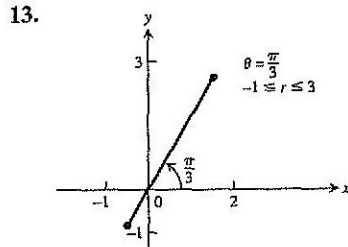
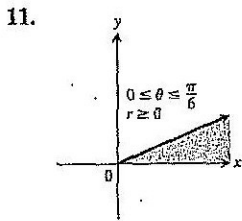
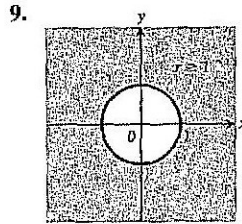
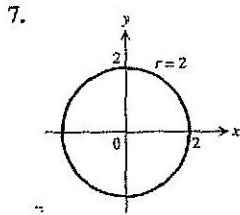
29. Show that the point  $(2, 3\pi/4)$  lies on the curve  $r = 2 \sin 2\theta$ .  
 30. Show that  $(1/2, 3\pi/2)$  lies on the curve  $r = -\sin(\theta/3)$ .

Find the points of intersection of the pairs of curves in Exercises 31–38.

31.  $r = 1 + \cos \theta$ ,  $r = 1 - \cos \theta$   
 32.  $r = 1 + \sin \theta$ ,  $r = 1 - \sin \theta$   
 33.  $r = 2 \sin \theta$ ,  $r = 2 \sin 2\theta$   
 34.  $r = \cos \theta$ ,  $r = 1 - \cos \theta$   
 35.  $r = \sqrt{2}$ ,  $r^2 = 4 \sin \theta$   
 36.  $r^2 = \sqrt{2} \sin \theta$ ,  $r^2 = \sqrt{2} \cos \theta$   
 37.  $r = 1$ ,  $r^2 = 2 \sin 2\theta$   
 38.  $r^2 = \sqrt{2} \cos 2\theta$ ,  $r^2 = \sqrt{2} \sin 2\theta$

 Find the points of intersection of the pairs of curves in Exercises 39–42.

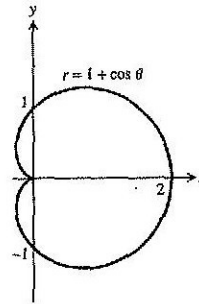
39.  $r^2 = \sin 2\theta$ ,  $r^2 = \cos 2\theta$   
 40.  $r = 1 + \cos \frac{\theta}{2}$ ,  $r = 1 - \sin \frac{\theta}{2}$   
 41.  $r = 1$ ,  $r = 2 \sin 2\theta$       42.  $r = 1$ ,  $r^2 = 2 \sin 2\theta$



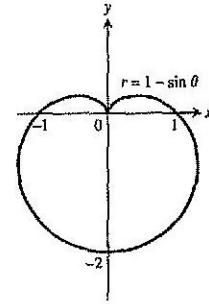
23.  $x = 2$ , vertical line through  $(2, 0)$   
 25.  $y = 0$ , the  $x$ -axis  
 27.  $y = 4$ , horizontal line through  $(0, 4)$   
 29.  $x + y = 1$ , line,  $m = -1$ ,  $b = 1$   
 31.  $x^2 + y^2 = 1$ , circle,  $C(0, 0)$ , radius 1  
 33.  $y - 2x = 5$ , line;  $m = 2$ ,  $b = 5$   
 35.  $y^2 = x$ , parabola, vertex  $(0, 0)$ , opens right  
 37.  $y = e^x$ , graph of natural exponential function  
 39.  $x + y = \pm 1$ , two straight lines of slope  $-1$ ,  $y$ -intercepts  $b = \pm 1$   
 41.  $(x + 2)^2 + y^2 = 4$ , circle,  $C(-2, 0)$ , radius 2  
 43.  $x^2 + (y - 4)^2 = 16$ , circle,  $C(0, 4)$ , radius 4  
 45.  $(x - 1)^2 + (y - 1)^2 = 2$ , circle,  $C(1, 1)$ , radius  $\sqrt{2}$   
 47.  $\sqrt{3}y + x = 4$     49.  $r \cos \theta = 7$     51.  $\theta = \pi/4$   
 53.  $r = 2$  or  $r = -2$     55.  $4r^2 \cos^2 \theta + 9r^2 \sin^2 \theta = 36$   
 57.  $r \sin^2 \theta = 4 \cos \theta$     59.  $r = 4 \sin \theta$   
 61.  $r^2 = 6r \cos \theta - 2r \sin \theta - 6$   
 63.  $(0, \theta)$ , where  $\theta$  is any angle

**Section 10.6, pp. 725–726**

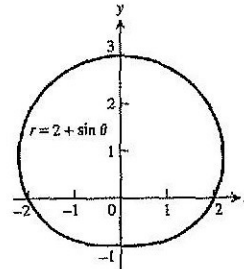
1.  $x$ -axis



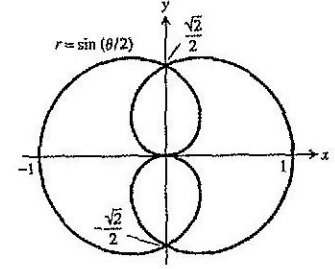
3.  $y$ -axis



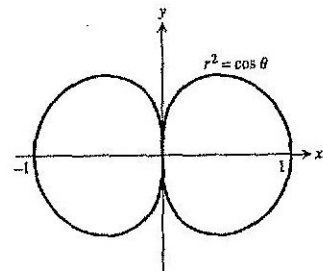
5.  $y$ -axis



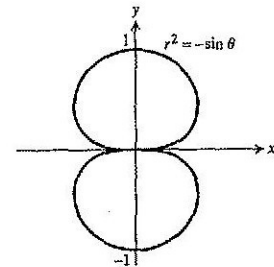
7.  $x$ -axis



9.  $x$ -axis,  $y$ -axis, origin

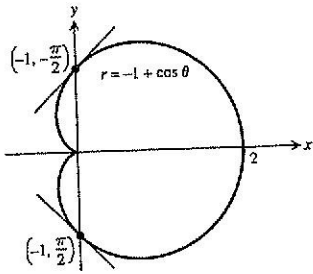


11.  $y$ -axis,  $x$ -axis, origin

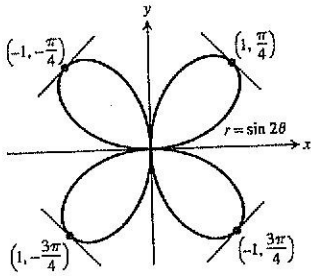


13.  $x$ -axis,  $y$ -axis, origin    15. Origin

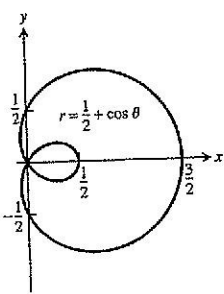
17. The slope at  $(-1, \pi/2)$  is  $-1$ , at  $(-1, -\pi/2)$  is  $1$ .



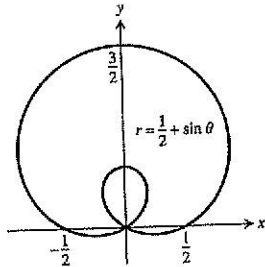
19. The slope at  $(1, \pi/4)$  is  $-1$ , at  $(-1, -\pi/4)$  is  $1$ , at  $(-1, 3\pi/4)$  is  $1$ , at  $(1, -3\pi/4)$  is  $-1$ .



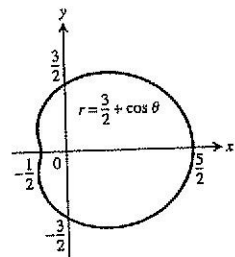
21. (a)



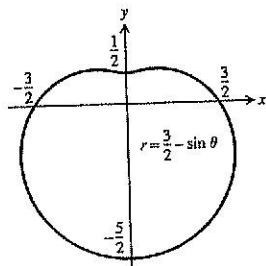
(b)



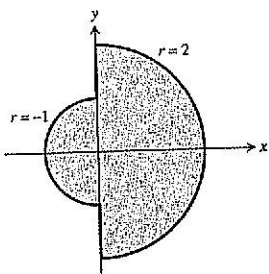
23. (a)



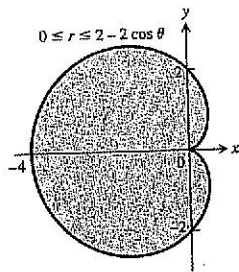
(b)



25.



27.



31.  $(0, 0), (1, \pi/2), (1, 3\pi/2)$

33.  $(0, 0), (\sqrt{3}, \pi/3), (-\sqrt{3}, -\pi/3)$

35.  $(\sqrt{2}, \pm\pi/6), (\sqrt{2}, \pm 5\pi/6)$

37.  $(1, \pi/12), (1, 5\pi/12), (1, 13\pi/12), (1, 17\pi/12)$

43. (a) 51.  $2y = \frac{2\sqrt{6}}{9}$

Section 10.7, pp. 730-731

1.  $18\pi$  3.  $\pi/8$  5. 2 7.  $\frac{\pi}{2} - 1$  9.  $5\pi - 8$

11.  $3\sqrt{3} - \pi$  13.  $\frac{\pi}{3} + \frac{\sqrt{3}}{2}$  15.  $12\pi - 9\sqrt{3}$

17. (a)  $\frac{3}{2} - \frac{\pi}{4}$  19.  $19/3$  21. 8

23.  $3(\sqrt{2} + \ln(1 + \sqrt{2}))$  25.  $\frac{\pi}{8} + \frac{3}{8}$  27.  $2\pi$

29.  $\pi\sqrt{2}$  31.  $2\pi(2 - \sqrt{2})$  37.  $(\frac{5}{6}a, 0)$

Section 10.8, pp. 737-739

1.  $r \cos(\theta - \pi/6) = 5, y = -\sqrt{3}x + 10$

3.  $r \cos(\theta - 4\pi/3) = 3, y = -(\sqrt{3}/3)x - 2\sqrt{3}$

5.  $y = 2 - x$  7.  $y = (\sqrt{3}/3)x + 2\sqrt{3}$

9.  $r \cos(\theta - \pi/4) = 3$  11.  $r \cos(\theta + \pi/2) = 5$

13.  $r = 8 \cos \theta$  15.  $r = 2\sqrt{2} \sin \theta$  17.  $C(2, 0), \text{radius} = 2$

19.  $C(1, \pi), \text{radius} = 1$  21.  $(x - 6)^2 + y^2 = 36, r = 12 \cos \theta$

23.  $x^2 + (y - 5)^2 = 25, r = 10 \sin \theta$

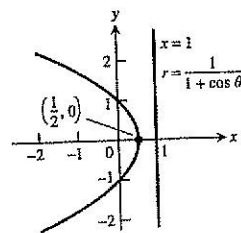
25.  $(x + 1)^2 + y^2 = 1, r = -2 \cos \theta$

27.  $x^2 + (y + 1/2)^2 = 1/4, r = -\sin \theta$  29.  $r = 2/(1 + \cos \theta)$

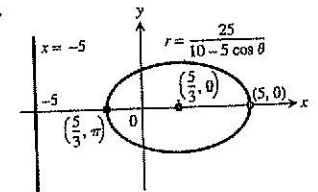
31.  $r = 30/(1 - 5 \sin \theta)$  33.  $r = 1/(2 + \cos \theta)$

35.  $r = 10/(5 - \sin \theta)$

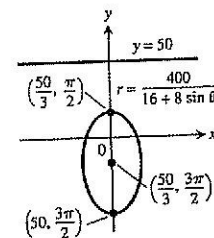
37.



39.



41.



43.

