Varberg [3.7] # 37

This problem asks when the speed of a particle is decreasing, if the particle's position is $s(t) = t^3 - 3t^2 - 24t - 6$. Speed is the magnitude of velocity; that is, speed $= |v(t)|$. For the particle in question,

$$\begin{aligned}|v(t)| &= \left|\frac{d}{dt}\left(t^3 - 3t^2 - 24t - 6\right)\right| \\ &= \left|3t^2 - 6t - 24\right|.\end{aligned}$$

Since we wish to know when the particle is slowing down, we ask when the time rate of change of $|v(t)|$ is negative. The easy way to calculate $\frac{d}{dt}|v(t)|$ is to use

$$\frac{d}{dt}|v(t)| = \frac{|v(t)|}{v(t)}\,\frac{d}{dt}(v(t)).$$

Doing so,

$$\begin{aligned}\frac{d}{dt}|v(t)| &= \frac{\left|3t^2 - 6t - 24\right|}{3t^2 - 6t - 24}\,\frac{d}{dt}(3t^2 - 6t - 24) \\ &= \frac{\left|3t^2 - 6t - 24\right|}{3t^2 - 6t - 24}\,(6t - 6).\end{aligned}$$

We seek the values of t for which $\frac{d}{dt}|v(t)| < 0$.

$$\frac{\left|3t^2 - 6t - 24\right|}{3t^2 - 6t - 24}\,(6t - 6) < 0$$

$$\frac{\left|t^2 - 2t - 12\right|}{(t-4)(t+2)}\,(t - 1) < 0.$$

The sign of the LHS is completely determined by

$$\frac{t-1}{(t-4)(t+2)}.$$

Analysis of the signs of $(t-1), (t-4)$, and $(t+2)$ using "split points" $x = -2, x = 1, x = 4$ shows that

$$\frac{t-1}{(t-4)(t+2)} < 0 \text{ when } x \in (-\infty, -2) \cup (1, 4).$$

So, the particle is slowing when $t < -2$ or when $1 < t < 4$. ■

An alternative solution would be the following.

$$|v(t)| = \left|3t^2 - 6t - 24\right| = \begin{cases} 3t^2 - 6t - 24, & \text{when } 3t^2 - 6t - 24 > 0 \\ -(3t^2 - 6t - 24), & \text{when } 3t^2 - 6t - 24 < 0. \end{cases}$$

Case 1.

$3t^2 - 6t - 24 > 0 \iff t^2 - 2t - 8 > 0 \iff (t-4)(t+2) > 0 \implies t < -2$ or $t > 4$.

On the other hand, $\frac{d}{dt}(3t^2 - 6t - 24) = 6t - 6$. Then $6t - 6 < 0 \implies t < 1$.

The values of t that satisfy the conditions $3t^2 - 6t - 24 > 0$ and $6t - 6 < 0$ are $(-\infty, -2) \cap (-\infty, 1) = (-\infty, -2)$.

Case 2.

$3t^2 - 6t - 24 < 0 \iff t^2 - 2t - 8 < 0 \iff (t-4)(t+2) < 0 \implies -2 < t < 4$.

On the other hand, $\frac{d}{dt}(-(3t^2 - 6t - 24)) = -6t + 6$. Then $-6t + 6 < 0 \implies t > 1$.

The values of t that satisfy the conditions $3t^2 - 6t - 24 < 0$ and $6t - 6 < 0$ are $(-2, 4) \cap (1, \infty) = (1, 4)$.

Conclusion: the particle is slowing when $t < -2$ or when $1 < t < 4$. ■