Comments on Exercise [2.5] #17

The purpose of this note is twofold.

(1) Point out where in my board working of Exercise [2.5] #17 I overlooked a simple fact that would have avoided the alternative path that we decided on.
(2) Mention that the alternative path resulted in a perfectly correct proof, in spite of it's looking different than the one in the *Student Solutions Manual*.

Prove
$$\lim_{x\to-1} x^2-2x-1=2.$$
Preliminary analysis (selecting δ).
$$\begin{aligned}\left|(x^2-2x-1)-2\right| &= \left|x^2-2x-3\right| \\ &= |x+1||x-3|.\end{aligned}$$
We need an upper bound for$|x-3|$. So we required $|x+1|<\delta\leq 1$. Then
$$\begin{aligned}|x+1| &\leq 1 \\ -1 \leq x+1 &\leq 1 \\ -5 \leq x-3 &\leq -3 \\ 3 \leq 3-x &\leq 5. \qquad (0.1)\end{aligned}$$
It was at this point that we gave up using 1 for an upper bound of $|x+1|$, because it lead to a bound for $|3-x|$ instead of $|x-3|$.

(1) What I might have done.

I should have realized that $|a-b|=|b-a|$. If I had, we would have continued,
$$\begin{aligned}|3-x| &\leq 5 \\ |x-3| &\leq 5.\end{aligned}$$
Then
$$\begin{aligned}|x+1||x-3| &\leq 5|x+1|. \\ 5|x+1| &< \varepsilon \\ |x+1| &< \frac{\varepsilon}{5}.\end{aligned}$$

Choose $\delta = \frac{\varepsilon}{5}$.

Proof. Fix $\varepsilon > 0$. Choose $\delta = \min\{1, \frac{\varepsilon}{5}\}$. Suppose $0 < |x+1| < \delta$. Then,

$$\begin{aligned} \left|(x^2 - 2x - 1) - 2\right| &= \left|x^2 - 2x - 3\right| \\ &= |x+1||x-3| \\ &\leq 5|x+1| \\ &< 5\delta \\ &= 5\left(\frac{\varepsilon}{5}\right) \\ &= \varepsilon. \end{aligned}$$

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(2) What I did.

At inequality 0.1, we decided to require δ to be 5. This lead to

$$\begin{aligned} |x+1| &\leq 5 \\ -5 \leq x+1 &\leq 5 \\ -9 \leq x-3 &\leq 1 \\ |x-3| &\leq 1. \end{aligned}$$

Then,

$$\begin{aligned} |x+1||x-3| &\leq 1 \cdot |x+1|. \\ 1 \cdot |x+1| &< \varepsilon \end{aligned}$$

Choose $\delta = \varepsilon$.

Proof. Fix $\varepsilon > 0$. Choose $\delta = \min\{5, \varepsilon\}\}$. Suppose $0 < |x+1| < \delta$. Then,

$$\begin{aligned} \left|(x^2 - 2x - 1) - 2\right| &= \left|x^2 - 2x - 3\right| \\ &= |x+1||x-3| \\ &\leq 1 \cdot |x+1| \\ &< \delta = \varepsilon. \end{aligned}$$

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