1.1 Simple cases

Prove that the limit at $x = 3$ of $\frac{2x^2 - 5x - 3}{x - 3}$ is 7; i.e. prove

$$\lim_{x \to 3} \frac{2x^2 - 5x - 3}{x - 3} = 7.$$

We must show the definition of limit is satisfied. For every value ϵ takes, we must compute a positive value for δ that guarantees $f(x)$ is within distance ϵ of 7 whenever x is within distance δ of 3, but not at 3. Obviously, the way we compute the value of δ must take into account the value of ϵ. The key to the proof is discovering how δ should depend on ϵ.

Preliminary analysis. The goal is to figure out how to compute δ, given ϵ, so that $0 < |x - 3| < \delta \implies |f(x) - 7| < \epsilon$. We reason as follows.

$$|f(x) - 7| = \left| \frac{2x^2 - 5x - 3}{x - 3} - 7 \right|$$

$$= \left| \frac{(2x + 1)(x - 3)}{x - 3} - 7 \right|$$

since $x \neq 3$ (remember, $0 < |x - 3|$)

$$= |(2x + 1) - 7|$$

$$= |2x - 6|$$

$$= |2(x - 3)|$$

$$= 2\,|x - 3|\,. \tag{1.1}$$

Since $|f(x) - 7| = 2\,|x - 3|\,,\ |f(x) - 7| < \epsilon \iff 2\,|x - 3| < \epsilon$. *And*

$$2\,|x - 3| < \epsilon \quad \text{whenever} \quad |x - 3| < \frac{\epsilon}{2}.$$

If for every value of ϵ, we choose $\delta = \frac{\epsilon}{2}$,

$$2\,|x - 3| < \epsilon \quad \text{whenever} \quad |x - 3| < \delta.$$

Proof. Fix $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{2}$. Suppose $0 < |x-3| < \delta$. Then

$$\begin{aligned}
|f(x) - 7| &= \left|\frac{2x^2 - 5x - 3}{x - 3} - 7\right| \\
&= \left|\frac{(2x+1)(x-3)}{x-3} - 7\right| \\
&= |(2x+1) - 7| \\
&= |2x - 6| \\
&= |2(x-3)| \\
&= 2\,|x-3| \\
&< 2\delta \\
&= 2\frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Therefore

$$\lim_{x \to 3} \frac{2x^2 - 5x - 3}{x - 3} = 7,$$

by definition of limit. □

Remark 1.1.

1. Be sure you can find the point in the proof at which we used the supposition $0 < |x-3| < \delta$.
2. The crucial step in the preliminary analysis is at equation 1.1. You will succeed at finding how to compute δ only if you make the expression that is to be bounded by δ appear.

1.2 Junk Factor

In the previous section, the key moment at which we knew we would succeed occurred at equation 1.1. Now let‘s consider an example in which when the path from the key moment to success is not as obvious.

Let $f(x) = x^2$. Prove that the limit at $x = 1$ of $f(x)$ is 1; i.e. prove

$$\lim_{x \to 1} x^2 = 1.$$

Preliminary analysis. Figure out how to compute δ, given ϵ, so that $0 < |x-1| < \delta \implies |f(x) - 1| < \epsilon$. Remember, we must make $|x-1|$ appear.

$$\begin{aligned}
|f(x) - 1| &= |x^2 - 1| \\
&= |x+1||x-1|.
\end{aligned}$$

Well, that was fast! $|x-1|$ is already on stage.

$$= |x+1||x-1| \tag{1.2}$$

The catch is that although $|x-1|$ appears, so does $|x+1|$. We can bound $|x-1|$ in the usual way, but what about $|x+1|$? It seems *that one* gets away on us. But, we can trap $|x+1|$, if we are a little bit clever. Suppose we require that in any case, δ is to be no less than 1.

Require $\delta \leq 1$ and $0 < |x-1| < \delta$. Then

$$|x-1| < \delta \leq 1$$
$$|x-1| < 1 \tag{1.3}$$
$$1 < x+1 < 3 \tag{1.4}$$

and this implies

$$|x+1| < 3. \tag{1.5}$$

Together equation 1.2 and equation 1.5 imply

$$|f(x)-1| < 3|x-1|.$$

A familiar place, indeed. If $3|x-1| < \epsilon$, $|f(x)-1|$ will be less than ϵ too.

$$3|x-1| < \epsilon$$
$$|x-1| < \frac{\epsilon}{3}.$$

We see that whenever $\delta = \frac{\epsilon}{3}$, $|f(x)-1| < |x+1||x-1| < 3|x-1| < 3\delta < 3\,\frac{\epsilon}{3} = \epsilon$. Our choice of δ, given any ϵ, will be either 1 or $\frac{\epsilon}{3}$ whichever is least.

Proof. Fix $\epsilon > 0$. Choose $\delta = \min\{1, \frac{\epsilon}{3}\}$. Suppose $0 < |x-1| < \delta$. Then

$$\begin{aligned} |f(x)-1| &= |x^2-1| \\ &= |x+1||x-1| \\ &< 3|x-1| \\ &< 3\delta \\ &< 3\,\frac{\epsilon}{3} \\ &< \epsilon \end{aligned}$$

Therefore

$$\lim_{x\to 1} x^2 = 1$$

by definition of limit. □

Remark 1.2. "Junk factor?"

We call the expression $|x+1|$ a junk factor, because it is easily bounded by a constant. In the above example, we see $|x+1||x-1|$ as $|\text{junk}||x-1|$.

Remark 1.3. "In the above example, how do we know to require $\delta \le 1$?"

The fact is, we could have picked *any* number. To see why, just work the analysis using a number other than 1.

Remark 1.4. "Why does inequality (1.3) imply inequality (1.5)?"

$|x-1| < 1$ says that x lies within 1 unit of 1. This places x in the interval $(0,2)$. This means that x can be up to 3 units away from -1; that is, $|x-(-1)| < 3$ which is more simply written as $|x+1| < 3$. Figure (1.1) may be helpful.

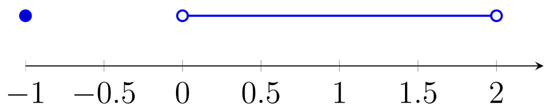

Figure 1.1

Example 1.1. Prove that $\lim_{x\to 3} \dfrac{1}{x+1} = \dfrac{1}{4}$.

Preliminary analysis.

$$\begin{aligned} \left|\frac{1}{x+1} - \frac{1}{4}\right| &= \left|\frac{3-x}{4x+4}\right| \\ &= |x-3| \cdot \frac{1}{4} \cdot \left|\frac{1}{x+1}\right|. \end{aligned}$$

Require $\delta \le 1$ and $0 < |x-3| < \delta$. This implies $-1 < x-3 < 1 \implies 3 < x+1 < 5$. So that

$$\frac{1}{5} < \frac{1}{x+1} < \frac{1}{3}.$$

Thus,

$$\begin{aligned} \left|\frac{1}{x+1} - \frac{1}{4}\right| &< \frac{1}{4} \cdot \frac{1}{3} \cdot |x-3| \\ &= \frac{1}{12} \cdot |x-3| \end{aligned}$$

This is less than ϵ when $|x-3| < 12\epsilon$.

$$\tag{1.6}$$

Therefore, given any ϵ the choice of $\delta = \min\{1, 12\epsilon\}$ will insure a successful proof.