

1.6. Building integers

In Section 1.2 on page 5 we said that the prime numbers are the building blocks of the integers. We considered how to decompose a composite number into its prime factors. Now we consider building up instead of tearing down. There is an obvious but helpful idea. If all the prime factors of an integer N are included in $\{p_1, p_2, p_3, \dots, p_n\}$, then the factors of a factor of N must belong to the set $\{p_1, p_2, p_3, \dots, p_n\}$.

For example, the prime factors of 30 are 2, 3, 5. If n is a divisor of 30, then n can have no more than one factor of 2, one factor of 3, and one factor of 5.

1.6.1. Counting factors

If we know the prime factors of a number, we can predict how many divisors the number has. The following examples show how.

Example 1.22

How many factors (prime and non-prime) has the number 30? (The prime factors of 30 are 2, 3, 5.)

Solution

Our project is to build a factor of 30. We do this as a sequence of tasks.

Task 1: Choose some number of 2s. There are two ways to do this. Choose no 2, $2^0 = 1$ or choose one 2, $2^1 = 2$. So, . . . 2 ways.

Task 2: Choose some number of 3s. There are two ways to do this. Choose no 3, $3^0 = 1$ or choose one 3, $3^1 = 3$. So, . . . 2 ways.

Task 3: Choose some number of 5s. There are two ways to do this. Choose no 5, $5^0 = 1$ or choose one 5, $5^1 = 5$. So, . . . 2 ways.

There are $2 \cdot 2 \cdot 2 = 6$ ways to complete the project. Therefore, there are 6 factors of 30.

Example 1.23

How many divisors has the number 504?

Solution

Note that $504 = 2^3 \cdot 3^2 \cdot 7^1$.

Project: Build a divisor of 504.

Task 1: Choose 0, 1, 2, or 3 factors of 2. . . . 4 ways.

Task 2: Choose 0, 1, or 2 factors of 3. . . . 3 ways.

Task 3: Choose 0 or 1 factor of 7. . . . 2 ways.

There are $4 \cdot 3 \cdot 2 = 24$ ways to complete the project. Therefore, there are 24 factors of 504.

1.6.2. Building divisors

Each divisor of a number must be a product of some combination of prime factors of that number. For example, 6 is a divisor of 12. The divisor 6 is built using one factor of 2 and one factor of 3. The divisor 1 is built using no factor of 2, $2^0 = 1$, and no factor of 3, $3^0 = 1$.

Example 1.24

Find all the divisors of 18.

Solution

The prime factorization of 18 is $2 \cdot 3^2$. We expect 6 divisors. Writing every combination of factors of 12,

$$2^0 \cdot 3^0 = 1$$

$$2^0 \cdot 3^1 = 3$$

$$2^0 \cdot 3^2 = 9$$

$$2^1 \cdot 3^0 = 2$$

$$2^1 \cdot 3^1 = 6$$

$$2^1 \cdot 3^2 = 18.$$

Example 1.25

Find all the factors of 100.

Solution

The prime factorization of 100 is $2^2 \cdot 5^2$. We expect 9 divisors. Writing every combination of factors of 100,

$$2^0 \cdot 5^0 = 1$$

$$2^1 \cdot 5^0 = 2$$

$$2^2 \cdot 5^0 = 4$$

$$2^0 \cdot 5^1 = 5$$

$$2^1 \cdot 5^1 = 10$$

$$2^2 \cdot 5^1 = 20$$

$$2^0 \cdot 5^2 = 25$$

$$2^1 \cdot 5^2 = 50$$

$$2^2 \cdot 5^2 = 100$$

Example 1.26

Find all the factors of 80.

Solution

The prime factorization of 40 is $2^4 \cdot 5^1$. We expect 10 divisors.

$$2^0 \cdot 5^0 = 1 \quad 2^1 \cdot 5^0 = 2 \quad 2^2 \cdot 5^0 = 4 \quad 2^3 \cdot 5^0 = 8 \quad 2^4 \cdot 5^0 = 16$$

$$2^0 \cdot 5^1 = 5 \quad 2^1 \cdot 5^1 = 10 \quad 2^2 \cdot 5^1 = 20 \quad 2^3 \cdot 5^1 = 40 \quad 2^4 \cdot 5^1 = 80$$

Exercise 1.5

How many divisors has each number

1. 72

3. 126

2. 40

4. 560

Write all divisors of the number

5. 8

7. 54

6. 98

8. 675
