Moreover, if a problem has been reduced to the integration of a rational function, it is then certain that an elementary primitive exists, even when the difficulty or impossibility of finding the factors of the denominator may preclude writing this primitive explicitly.

PROBLEMS

1. This problem contains some integrals which require little more than algebraic manipulation, and consequently test your ability to discover algebraic tricks, rather than your understanding of the integration processes. Nevertheless, any one of these tricks might be an important preliminary step in an honest integration problem. Moreover, you want to have some feel for which integrals are easy, so that you can see when the end of an integration process is in sight. The answer section, if you resort to it, will only reveal what algebra you should have used.

(i) $\displaystyle\int \frac{\sqrt[5]{x^3} + \sqrt[6]{x}}{\sqrt{x}}\, dx.$

(ii) $\displaystyle\int \frac{dx}{\sqrt{x-1} + \sqrt{x+1}}.$

(iii) $\displaystyle\int \frac{e^x + e^{2x} + e^{3x}}{e^{4x}}\, dx.$

(iv) $\displaystyle\int \frac{a^x}{b^x}\, dx.$

(v) $\displaystyle\int \tan^2 x\, dx.$ (Trigonometric integrals are always very touchy, because there are so many trigonometric identities that an easy problem can easily look hard.)

(vi) $\displaystyle\int \frac{dx}{a^2 + x^2}.$

(vii) $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}}.$

(viii) $\displaystyle\int \frac{dx}{1 + \sin x}.$

(ix) $\displaystyle\int \frac{8x^2 + 6x + 4}{x+1}\, dx.$

(x) $\displaystyle\int \frac{1}{\sqrt{2x - x^2}}\, dx.$

2. The following integrations involve simple substitutions, most of which you should be able to do in your head.

(i) $\displaystyle\int e^x \sin e^x\, dx.$

(ii) $\int xe^{-x^2}\,dx.$

(iii) $\int \frac{\log x}{x}\,dx.$ (In the text this was done by parts.)

(iv) $\int \frac{e^x\,dx}{e^{2x}+2e^x+1}.$

(v) $\int e^{e^x}e^x\,dx.$

(vi) $\int \frac{x\,dx}{\sqrt{1-x^4}}.$

(vii) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx.$

(viii) $\int x\sqrt{1-x^2}\,dx.$

(ix) $\int \log(\cos x)\tan x\,dx.$

(x) $\int \frac{\log(\log x)}{x\log x}\,dx.$

3. Integration by parts.

(i) $\int x^2e^x\,dx.$

(ii) $\int x^3e^{x^2}\,dx.$

(iii) $\int e^{ax}\sin bx\,dx.$

(iv) $\int x^2\sin x\,dx.$

(v) $\int (\log x)^3\,dx.$

(vi) $\int \frac{\log(\log x)}{x}\,dx.$

(vii) $\int \sec^3 x\,dx.$ (This is a tricky and important integral that often comes up. If you do not succeed in evaluating it, be sure to consult the answers.)

(viii) $\int \cos(\log x)\,dx.$

(ix) $\int \sqrt{x}\log x\,dx.$

(x) $\int x(\log x)^2\,dx.$

4. The following integrations can all be done with substitutions of the form $x = \sin u$, $x = \cos u$, etc. To do some of these you will need to remember that

$$\int \sec x \, dx = \log(\sec x + \tan x)$$

as well as the following formula, which can also be checked by differentiation:

$$\int \csc x \, dx = -\log(\csc x + \cot x).$$

In addition, at this point the derivatives of all the trigonometric functions should be kept handy.

(i) $\displaystyle\int \frac{dx}{\sqrt{1-x^2}}$. (You already know this integral, but use the substitution $x = \sin u$ anyway, just to see how it works out.)

(ii) $\displaystyle\int \frac{dx}{\sqrt{1+x^2}}$. (Since $\tan^2 u + 1 = \sec^2 u$, you want to use the substitution $x = \tan u$.)

(iii) $\displaystyle\int \frac{dx}{\sqrt{x^2-1}}$.

(iv) $\displaystyle\int \frac{dx}{x\sqrt{x^2-1}}$. (The answer will be a certain inverse function that was given short shrift in the text.)

(v) $\displaystyle\int \frac{dx}{x\sqrt{1-x^2}}$.

(vi) $\displaystyle\int \frac{dx}{x\sqrt{1+x^2}}$.

(vii) $\displaystyle\int x^3\sqrt{1-x^2}\,dx$.

(viii) $\displaystyle\int \sqrt{1-x^2}\,dx$.

You will need to remember the methods for integrating powers of sin and cos.

(ix) $\displaystyle\int \sqrt{1+x^2}\,dx$.

(x) $\displaystyle\int \sqrt{x^2-1}\,dx$.

5. The following integrations involve substitutions of various types. There is no substitute for cleverness, but there is a general rule to follow: substitute for an expression which appears frequently or prominently; if two different troublesome expressions appear, try to express them both in terms of some new expression. And don't forget that it usually helps to express x directly in terms of u, to find out the proper expression to substitute for dx.

(i) $\displaystyle\int \frac{dx}{1+\sqrt{x+1}}$.

(ii) $\displaystyle\int \frac{dx}{1+e^x}$.

(iii) $\displaystyle\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$.

(iv) $\displaystyle\int \frac{dx}{\sqrt{1+e^x}}$. (The substitution $u = e^x$ leads to an integral requiring yet another substitution; this is all right, but both substitutions can be done at once.)

(v) $\displaystyle\int \frac{dx}{2+\tan x}$.

(vi) $\displaystyle\int \frac{dx}{\sqrt{\sqrt{x}+1}}$. (Another place where one substitution can be made to do the work of two.)

(vii) $\displaystyle\int \frac{4^x+1}{2^x+1}\,dx$.

(viii) $\displaystyle\int e^{\sqrt{x}}\,dx$.

(ix) $\displaystyle\int \frac{\sqrt{1-x}}{1-\sqrt{x}}\,dx$. (In this case two successive substitutions work out best; there are two obvious candidates for the first substitution, and either will work.)

*(x) $\displaystyle\int \sqrt{\frac{x-1}{x+1}}\cdot\frac{1}{x^2}\,dx$.

6. The previous problem provided gratis a haphazard selection of rational functions to be integrated. Here is a more systematic selection.

(i) $\displaystyle\int \frac{2x^2+7x-1}{x^3+x^2-x-1}\,dx$.

(ii) $\displaystyle\int \frac{2x+1}{x^3-3x^2+3x-1}\,dx$.

(iii) $\displaystyle\int \frac{x^3+7x^2-5x+5}{(x-1)^2(x+1)^3}\,dx$.

(iv) $\displaystyle\int \frac{2x^2+x+1}{(x+3)(x-1)^2}\,dx$.

(v) $\displaystyle\int \frac{x+4}{x^2+1}\,dx$.

(vi) $\displaystyle\int \frac{x^3+x+2}{x^4+2x^2+1}\,dx$.

(vii) $\displaystyle\int \frac{3x^2+3x+1}{x^3+2x^2+2x+1}\,dx$.

(viii) $\displaystyle\int \frac{dx}{x^4+1}$.

(ix) $\displaystyle\int \frac{2x}{(x^2+x+1)^2}\,dx$.

(x) $\displaystyle\int \frac{3x}{(x^2+x+1)^3}\,dx$.

***7.** Potpourri. (No holds barred.) The following integrations involve all the methods of the previous problems

(i) $\displaystyle\int \frac{\arctan x}{1+x^2}\,dx.$

(ii) $\displaystyle\int \frac{x \arctan x}{(1+x^2)^3}\,dx.$

(iii) $\displaystyle\int \log\sqrt{1+x^2}\,dx.$

(iv) $\displaystyle\int x\log\sqrt{1+x^2}\,dx.$

(v) $\displaystyle\int \frac{x^2-1}{x^2+1}\cdot\frac{1}{\sqrt{1+x^4}}\,dx.$

(vi) $\displaystyle\int \arcsin\sqrt{x}\,dx.$

(vii) $\displaystyle\int \frac{x}{1+\sin x}\,dx.$

(viii) $\displaystyle\int e^{\sin x}\cdot\frac{x\cos^3 x-\sin x}{\cos^2 x}\,dx.$

(ix) $\displaystyle\int \sqrt{\tan x}\,dx.$

(x) $\displaystyle\int \frac{dx}{x^6+1}.$ (To factor x^6+1, first factor y^3+1, using Problem 1-1.)

The following two problems provide still more practice at integration, if you need it (and can bear it). Problem 8 involves algebraic and trigonometric manipulations and integration by parts, while Problem 9 involves substitutions. (Of course, in many cases the resulting integrals will require still further manipulations.)

8. Find the following integrals.

(i) $\displaystyle\int \log(a^2+x^2)\,dx.$

(ii) $\displaystyle\int \frac{1+\cos x}{\sin^2 x}\,dx.$

(iii) $\displaystyle\int \frac{x+1}{\sqrt{4-x^2}}\,dx.$

(iv) $\displaystyle\int x\arctan x\,dx.$

(v) $\displaystyle\int \sin^3 x\,dx.$

(vi) $\displaystyle\int \frac{\sin^3 x}{\cos^2 x}\,dx.$

(vii) $\int x^2 \arctan x \, dx.$

(viii) $\int \frac{x\,dx}{\sqrt{x^2-2x+2}}.$

(ix) $\int \sec^3 x \tan x \, dx.$

(x) $\int x \tan^2 x \, dx.$

9. Find the following integrals.

(i) $\int \frac{dx}{(a^2+x^2)^2}.$

(ii) $\int \sqrt{1-\sin x}\, dx.$

(iii) $\int \arctan \sqrt{x}\, dx.$

(iv) $\int \sin \sqrt{x+1}\, dx.$

(v) $\int \frac{\sqrt{x^3-2}}{x}\, dx.$

(vi) $\int \log(x+\sqrt{x^2-1})\, dx.$

(vii) $\int \log(x+\sqrt{x})\, dx.$

(viii) $\int \frac{dx}{x-x^{3/5}}.$

(ix) $\int (\arcsin x)^2\, dx.$

(x) $\int x^5 \arctan(x^2)\, dx.$

10. If you have done Problem 18-9, the integrals (ii) and (iii) in Problem 4 will look very familiar. In general, the substitution $x = \cosh u$ often works for integrals involving $\sqrt{x^2-1}$, while $x = \sinh u$ is the thing to try for integrals involving $\sqrt{x^2+1}$. Try these substitutions on the other integrals in Problem 4. (The method is not really recommended; it is easier to stick with trigonometric substitutions.)

***11.** The world's sneakiest substitution is undoubtedly

$$t = \tan\frac{x}{2}, \quad x = 2\arctan t,$$
$$dx = \frac{2}{1+t^2}\,dt.$$

As we found in Problem 15-17, this substitution leads to the expressions

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}.$$

This substitution thus transforms any integral which involves only sin and cos, combined by addition, multiplication, and division, into the integral of a rational function. Find

(i) $\int \frac{dx}{1+\sin x}$. (Compare your answer with Problem 1(viii).)

(ii) $\int \frac{dx}{1-\sin^2 x}$. (In this case it is better to let $t = \tan x$. Why?)

(iii) $\int \frac{dx}{a\sin x + b\cos x}$. (There is also another way to do this, using Problem 15-8.)

(iv) $\int \sin^2 x\,dx$. (An exercise to convince you that this substitution should be used only as a last resort.)

(v) $\int \frac{dx}{3+5\sin x}$. (A last resort.)

***12.** Derive the formula for $\int \sec x\,dx$ in the following two ways:

(a) By writing

$$\begin{aligned}\frac{1}{\cos x} &= \frac{\cos x}{\cos^2 x}\\ &= \frac{\cos x}{1-\sin^2 x}\\ &= \frac{1}{2}\left[\frac{\cos x}{1+\sin x} + \frac{\cos x}{1-\sin x}\right],\end{aligned}$$

an expression obviously inspired by partial fraction decompositions. Be sure to note that $\int \cos x/(1-\sin x)\,dx = -\log(1-\sin x)$; the minus sign is very important. And remember that $\frac{1}{2}\log\alpha = \log\sqrt{\alpha}$. From there on, keep doing algebra, and trust to luck.

(b) By using the substitution $t = \tan x/2$. One again, quite a bit of manipulation is required to put the answer in the desired form; the expression $\tan x/2$ can be attacked by using Problem 15-9, or both answers can be expressed in terms of t. There is another expression for $\int \sec x\,dx$, which is less cumbersome than $\log(\sec x + \tan x)$; using Problem 15-9, we obtain

$$\int \sec x\,dx = \log\left(\frac{1+\tan\dfrac{x}{2}}{1-\tan\dfrac{x}{2}}\right) = \log\left(\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right).$$

This last expression was actually the one first discovered, and was due, not to any mathematician's cleverness, but to a curious historical accident: In 1599 Wright computed nautical tables that amounted to definite integrals of sec. When the first tables for the logarithms of tangents were produced, the correspondence between the two tables was immediately noticed (but remained unexplained until the invention of calculus).

13. The derivation of $\int e^x \sin x\,dx$ given in the text seems to prove that the only primitive of $f(x) = e^x \sin x$ is $F(x) = e^x(\sin x - \cos x)/2$, whereas $F(x) = e^x(\sin x - \cos x)/2 + C$ is also a primitive for any number C. Where does C come from? (What is the meaning of the equation

$$\int e^x \sin x\,dx = e^x \sin x - e^x \cos x - \int e^x \sin x\,dx?)$$

14. Suppose that f'' is continuous and that

$$\int_0^\pi [f(x) + f''(x)] \sin x\,dx = 2.$$

Given that $f(\pi) = 1$, compute $f(0)$.

15. (a) Find $\int \arcsin x\,dx$, using the same trick that worked for log and arctan.
*(b) Generalize this trick: Find $\int f^{-1}(x)\,dx$ in terms of $\int f(x)\,dx$. Compare with Problems 12-18 and 14-17.

16. (a) Find $\int \sin^4 x\,dx$ in two different ways: first using the reduction formula, and then using the formula for $\sin^2 x$.
(b) Combine your answers to obtain an impressive trigonometric identity.

17. Express $\int \log(\log x)\,dx$ in terms of $\int (\log x)^{-1}\,dx$. (Neither is expressible in terms of elementary functions.)

18. Express $\int x^2 e^{-x^2}\,dx$ in terms of $\int e^{-x^2}\,dx$.

19. Prove that the function $f(x) = e^x/(e^{5x} + e^x + 1)$ has an elementary primitive. (Do not try to find it!)

20. Prove the reduction formulas in the text. For the third one write

$$\int \frac{dx}{(1+x^2)^n} = \int \frac{dx}{(1+x^2)^{n-1}} - \int \frac{x^2\,dx}{(1+x^2)^n}$$

and work on the last integral. (Another possibility is to use the substitution $x = \tan u$.)

21. Find a reduction formula for

(a) $\displaystyle\int x^n e^x\,dx$

(b) $\displaystyle\int (\log x)^n\,dx$.

***22.** Prove that

$$\int_1^{\cosh x} \sqrt{t^2 - 1}\,dt = \frac{\cosh x \sinh x}{2} - \frac{x}{2}.$$

(See Problem 18-6 for the significance of this computation.)

23. Prove that

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx.$$

(A geometric interpretation makes this clear, but it is also a good exercise in the handling of limits of integration during a substitution.)

24. Prove that the area of a circle of radius r is πr^2. (Naturally you must remember that π is defined as the area of the unit circle.)

25. Let ϕ be a nonnegative integrable function such that $\phi(x) = 0$ for $|x| \geq 1$ and such that $\int_{-1}^{1} \phi = 1$. For $h > 0$, let

$$\phi_h(x) = \frac{1}{h}\phi(x/h).$$

(a) Show that $\phi_h(x) = 0$ for $|x| \geq h$ and that $\int_{-h}^{h} \phi_h = 1$.

(b) Let f be integrable on $[-1, 1]$ and continuous at 0. Show that

$$\lim_{h\to 0^+} \int_{-1}^{1} \phi_h f = \lim_{h\to 0^+} \int_{-h}^{h} \phi_h f = f(0).$$

(c) Show that

$$\lim_{h\to 0^+} \int_{-1}^{1} \frac{h}{h^2+x^2}\,dx = \pi.$$

The final part of this problem might appear, at first sight, to be an exact analogue of part (b), but it actually requires more careful argument.

(d) Let f be integrable on $[-1, 1]$ and continuous at 0. Show that

$$\lim_{h\to 0^+} \int_{-1}^{1} \frac{h}{h^2+x^2} f(x)\,dx = \pi f(0).$$

Hint: If h is small, then $h/(h^2+x^2)$ will be small on most of $[-1, 1]$.

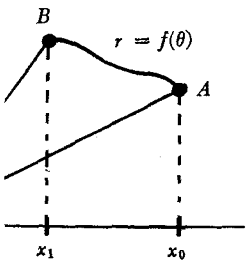
The next two problems use the formula

$$\frac{1}{2}\int_{\theta_0}^{\theta_1} f(\theta)^2\,d\theta,$$

derived in Problem 13-24, for the area of a region bounded by the graph of f in polar coordinates.

26. For each of the following functions, find the area bounded by the graphs in polar coordinates. (Be careful about the proper range for θ, or you will get nonsensical results!)

(i) $f(\theta) = a\sin\theta$.
(ii) $f(\theta) = 2 + \cos\theta$.
(iii) $f(\theta)^2 = 2a^2\cos 2\theta$.
(iv) $f(\theta) = a\cos 2\theta$.



27. Figure 1 shows the graph of f in polar coordinates; the region OAB thus has area $\frac{1}{2}\int_{\theta_0}^{\theta_1} f(\theta)^2\,d\theta$. Now suppose that this graph also happens to be the ordinary graph of some function g. Then the region OAB also has area

$$\text{area } \Delta O x_1 B + \int_{x_1}^{x_0} g - \text{ area } \Delta O x_0 A.$$

Prove analytically that these two numbers are indeed the same. Hint: The function g is determined by the equations

$$x = f(\theta)\cos\theta, \qquad g(x) = f(\theta)\sin\theta.$$

The next four problems use the formulas, derived in Problems 3 and 4 of the Appendix to Chapter 13, for the length of a curve represented parametrically (and, in particular, as the graph of a function in polar coordinates).

28. Let c be a curve represented parametrically by u and v on $[a, b]$, and let h be an increasing function with $h(\bar{a}) = a$ and $h(\bar{b}) = b$. Then on $[\bar{a}, \bar{b}]$ the functions $\bar{u} = u \circ h$, $\bar{v} = v \circ h$ give a parametric representation of another curve $\bar{c}$; intuitively, $\bar{c}$ is just the same curve c traversed at a different rate.

(a) Show, directly from the definition of length, that the length of c on $[a, b]$ equals the length of $\bar{c}$ on $[\bar{a}, \bar{b}]$.
(b) Assuming differentiability of any functions required, show that the lengths are equal by using the integral formula for length, and the appropriate substitution.

29. Find the length of the following curves, all described as the graphs of functions, except for (iii), which is represented parametrically.

(i) $f(x) = \frac{1}{3}(x^2 + 2)^{3/2}, \quad 0 \le x \le 1.$
(ii) $f(x) = x^3 + \frac{1}{12x}, \quad 1 \le x \le 2.$
(iii) $x = a^3\cos^3 t, \quad y = a^3\sin^3 t, \quad 0 \le t \le 2\pi.$
(iv) $f(x) = \log(\cos x), \quad 0 \le x \le \pi/6.$
(v) $f(x) = \log x, \quad 1 \le x \le e.$
(vi) $f(x) = \arcsin e^x, \quad -\log 2 \le x \le 0.$

30. For the following functions, find the length of the graph in polar coordinates.

(i) $f(\theta) = a\cos\theta.$
(ii) $f(\theta) = a(1 - \cos\theta).$
(iii) $f(\theta) = a\sin^2\theta/2.$
(iv) $f(\theta) = \theta \quad 0 \le \theta \le 2\pi.$
(v) $f(\theta) = 3\sec\theta \quad 0 \le \theta \le \pi/3.$

31. In Problem 8 of the Appendix to Chapter 12 we described the cycloid, which has the parametric representation

$$x = u(t) = a(t - \sin t), \qquad y = v(t) = a(1 - \cos t).$$

(a) Find the length of one arch of the cycloid. [Answer: $8a$.]

(b) Recall that the cycloid is the graph of $v \circ u^{-1}$. Find the area under one arch of the cycloid by using the appropriate substitution in $\int f$ and evaluating the resultant integral. [Answer: $3\pi a^2$.]

32. Use induction and integration by parts to generalize Problem 14-13:

$$\int_0^x \frac{f(u)(x-u)^n}{n!}\,du = \int_0^x \left(\int_0^{u_n} \left(\dots\left(\int_0^{u_1} f(t)\,dt\right) du_1\right)\dots\right) du_n.$$

33. If f' is continuous on $[a, b]$, use integration by parts to prove the Riemann-Lebesgue Lemma for f:

$$\lim_{\lambda\to\infty} \int_a^b f(t)\sin(\lambda t)\,dt = 0.$$

This result is just a special case of Problem 15-26, but it can be used to prove the general case (in much the same way that the Riemann-Lebesgue Lemma was derived in Problem 15-26 from the special case in which f is a step function).

34. The Mean Value Theorem for Integrals was introduced in Problem 13-23. The "Second Mean Value Theorem for Integrals" states the following. Suppose that f is integrable on $[a, b]$ and that ϕ is either nondecreasing or nonincreasing on $[a, b]$. Then there is a number ξ in $[a, b]$ such that

$$\int_a^b f(x)\phi(x)\,dx = \phi(a)\int_a^{\xi} f(x)\,dx + \phi(b)\int_{\xi}^b f(x)\,dx.$$

In this problem, we will assume that f is continuous and that ϕ is differentiable, with a continuous derivative ϕ'.

(a) Prove that if the result is true for nonincreasing ϕ, then it is also true for nondecreasing ϕ.

(b) Prove that if the result is true for nonincreasing ϕ satisfying $\phi(b) = 0$, then it is true for all nonincreasing ϕ.

Thus, we can assume that ϕ is nonincreasing and $\phi(b) = 0$. In this case, we have to prove that

$$\int_a^b f(x)\phi(x) = \phi(a)\int_a^{\xi} f(x)\,dx.$$

(c) Prove this by using integration by parts.

(d) Show that the hypothesis that ϕ is either nondecreasing or nonincreasing is needed.

From this special case of the Second Mean Value Theorem for Integrals, the general case could be derived by some approximation arguments, just as in the case of the Riemann-Lebesgue Lemma. But there is a more instructive way, outlined in the next problem.

35. (a) Given $a_1, \dots, a_n$ and $b_1, \dots, b_n$, let $s_k = a_1 + \cdots + a_k$. Show that

$$(*) \quad a_1b_1 + \cdots + a_nb_n = s_1(b_1 - b_2) + s_2(b_2 - b_3) + \cdots + s_{n-1}(b_{n-1} - b_n) + s_nb_n$$

This disarmingly simple formula is sometimes called "Abel's formula for summation by parts." It may be regarded as an analogue for sums of the integration by parts formula

$$\int_a^b f'(x)g(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)\,dx,$$

especially if we use Riemann sums (Chapter 13, Appendix). In fact, for a partition $P = \{t_0, \dots, t_n\}$ of $[a, b]$, the left side is approximately

$$(1) \qquad \sum_{k=1}^n f'(t_k)g(t_{k-1})(t_k - t_{k-1}),$$

while the right side is approximately

$$f(b)g(b) - f(a)g(a) - \sum_{k=1}^n f(t_k)g'(t_k)(t_k - t_{k-1})$$

which is approximately

$$\begin{aligned} f(b)g(b) - f(a)g(a) &- \sum_{k=1}^n f(t_k)\frac{g(t_k) - g(t_{k-1})}{t_k - t_{k-1}}(t_k - t_{k-1}) \\ &= f(b)g(b) - f(a)g(a) + \sum_{k=1}^n f(t_k)[g(t_{k-1}) - g(t_k)] \\ &= f(b)g(b) - f(a)g(a) + \sum_{k=1}^n [f(t_k) - f(a)] \cdot [g(t_{k-1}) - g(t_k)] \\ &\qquad + f(a)\sum_{k=1}^n g(t_{k-1}) - g(t_k). \end{aligned}$$

Since the right-most sum is just $g(a) - g(b)$, this works out to be

$$(2) \qquad [f(b) - f(a)]g(b) + \sum_{k=1}^n [f(t_k) - f(a)] \cdot [g(t_{k-1}) - g(t_k)].$$

If we choose

$$a_k = f'(t_k)(t_k - t_{k-1}), \qquad b_k = g(t_{k-1})$$

then

$$(1) \qquad \text{is} \qquad \sum_{k=1}^n a_kb_k,$$

which is the left side of $(*)$, while

$$s_k = \sum_{i=1}^k f'(t_i)(t_i - t_{i-1}) \quad \text{is approximately} \quad \sum_{i=1}^k f(t_i) - f(t_{i-1}) = f(t_k) - f(a),$$

so

$$(2) \qquad \text{is approximately} \qquad s_nb_n + \sum_{k=1}^n s_k(b_k - b_{k-1}),$$

which is the right side of $(*)$.

This discussion is not meant to suggest that Abel's formula can actually be derived from the formula for integration by parts, or *vice versa*. But, as we shall see, Abel's formula can often be used as a substitute for integration by parts in situations where the functions in question aren't differentiable.

(b) Suppose that $\{b_n\}$ is nonincreasing, with $b_n \geq 0$ for each n, and that

$$m \leq a_1 + \cdots + a_n \leq M$$

for all n. Prove Abel's Lemma:

$$b_1 m \leq a_1 b_1 + \cdots + a_n b_n \leq b_1 M.$$

(And, moreover,

$$b_k m \leq a_k b_k + \cdots + a_n b_n \leq b_k M,$$

a formula which only looks more general, but really isn't.)

(c) Let f be integrable on $[a, b]$ and let ϕ be nonincreasing on $[a, b]$ with $\phi(b) = 0$. Let $P = \{t_0, \dots, t_n\}$ be a partition of $[a, b]$. Show that the sum

$$\sum_{i=1}^{n} f(t_{i-1})\phi(t_{i-1})(t_i - t_{i-1})$$

lies between the smallest and the largest of the sums

$$\phi(a) \sum_{i=1}^{k} f(t_{i-1})(t_i - t_{i-1}).$$

Conclude that

$$\int_a^b f(x)\phi(x)\,dx$$

lies between the minimum and the maximum of

$$\phi(a) \int_a^x f(t)\,dt,$$

and that it therefore equals $\phi(a) \int_a^{\xi} f(t)\,dt$ for some ξ in $[a, b]$.

36. (a) Show that the following improper integrals both converge.

(i) $\displaystyle\int_0^1 \sin\left(x + \frac{1}{x}\right) dx.$

(ii) $\displaystyle\int_0^1 \sin^2\left(x + \frac{1}{x}\right) dx.$

(b) Decide which of the following improper integrals converge.

(i) $\displaystyle\int_1^\infty \sin\left(\frac{1}{x}\right) dx.$

(ii) $\displaystyle\int_1^\infty \sin^2\left(\frac{1}{x}\right) dx.$

37. (a) Compute the (improper) integral $\int_0^1 \log x \, dx$.

(b) Show that the improper integral $\int_0^\pi \log(\sin x) \, dx$ converges.

(c) Use the substitution $x = 2u$ to show that

$$\int_0^\pi \log(\sin x) \, dx = 2 \int_0^{\pi/2} \log(\sin x) \, dx + 2 \int_0^{\pi/2} \log(\cos x) \, dx + \pi \log 2.$$

(d) Compute $\int_0^{\pi/2} \log(\cos x) \, dx$.

(e) Using the relation $\cos x = \sin(\pi/2 - x)$, compute $\int_0^\pi \log(\sin x) \, dx$.

38. Prove the following version of integration by parts for improper integrals:

$$\int_a^\infty u'(x)v(x) \, dx = u(x)v(x) \Big|_a^\infty - \int_a^\infty u(x)v'(x) \, dx.$$

The first symbol on the right side means, of course,

$$\lim_{x \to \infty} u(x)v(x) - u(a)v(a).$$

***39.** One of the most important functions in analysis is the gamma function,

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \, dt.$$

(a) Prove that the improper integral $\Gamma(x)$ is defined if $x > 0$.

(b) Use integration by parts (more precisely, the improper integral version in the previous problem) to prove that

$$\Gamma(x + 1) = x\Gamma(x).$$

(c) Show that $\Gamma(1) = 1$, and conclude that $\Gamma(n) = (n - 1)!$ for all natural numbers n.

The gamma function thus provides a simple example of a continuous function which "interpolates" the values of $n!$ for natural numbers n. Of course there are infinitely many continuous functions f with $f(n) = (n - 1)!$; there are even infinitely many continuous functions f with $f(x + 1) = xf(x)$ for all $x > 0$. However, the gamma function has the important additional property that $\log \circ \Gamma$ is convex, a condition which expresses the extreme smoothness of this function. A beautiful theorem due to Harold Bohr and Johannes Mollerup states that Γ is the only function f with $\log \circ f$ convex, $f(1) = 1$ and $f(x + 1) = xf(x)$. See the Suggested Reading for a reference.

***40.** (a) Use the reduction formula for $\int \sin^n x \, dx$ to show that

$$\int_0^{\pi/2} \sin^n x \, dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x \, dx.$$

(b) Now show that

$$\int_0^{\pi/2} \sin^{2n+1} x\,dx = \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot\ldots\cdot\frac{2n}{2n+1},$$

$$\int_0^{\pi/2} \sin^{2n} x\,dx = \frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\ldots\cdot\frac{2n-1}{2n},$$

and conclude that

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\ldots\cdot\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\frac{\displaystyle\int_0^{\pi/2} \sin^{2n} x\,dx}{\displaystyle\int_0^{\pi/2} \sin^{2n+1} x\,dx}.$$

(c) Show that the quotient of the two integrals in this expression is between 1 and $1 + 1/2n$, starting with the inequalities

$$0 < \sin^{2n+1} x \le \sin^{2n} x \le \sin^{2n-1} x \quad \text{for } 0 < x < \pi/2.$$

This result, which shows that the products

$$\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\ldots\cdot\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}$$

can be made as close to $\pi/2$ as desired, is usually written as an infinite product, known as Wallis' product:

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\ldots.$$

(d) Show also that the products

$$\frac{1}{\sqrt{n}}\,\frac{2\cdot4\cdot6\cdot\ldots\cdot2n}{1\cdot3\cdot5\cdot\ldots\cdot(2n-1)}$$

can be made as close to $\sqrt{\pi}$ as desired. (This fact is used in the next problem and in Problem 27-19.)

****41.** It is an astonishing fact that improper integrals $\int_0^\infty f(x)\,dx$ can often be computed in cases where ordinary integrals $\int_a^b f(x)\,dx$ cannot. There is no elementary formula for $\int_a^b e^{-x^2}\,dx$, but we can find the value of $\int_0^\infty e^{-x^2}\,dx$ precisely! There are many ways of evaluating this integral, but most require some advanced techniques; the following method involves a fair amount of work, but no facts that you do not already know.

(a) Show that

$$\int_0^1 (1-x^2)^n\,dx = \frac{2}{3}\cdot\frac{4}{5}\cdot\ldots\cdot\frac{2n}{2n+1},$$

$$\int_0^\infty \frac{1}{(1+x^2)^n}\,dx = \frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\ldots\cdot\frac{2n-3}{2n-2}.$$

(This can be done using reduction formulas, or by appropriate substitutions, combined with the previous problem.)

(b) Prove, using the derivative, that

$$1-x^2 \le e^{-x^2} \qquad \text{for } 0 \le x \le 1.$$

$$e^{-x^2} \le \frac{1}{1+x^2} \qquad \text{for } 0 \le x.$$

(c) Integrate the nth powers of these inequalities from 0 to 1 and from 0 to ∞, respectively. Then use the substitution $y = \sqrt{n}\,x$ to show that

$$\sqrt{n}\frac{2}{3}\cdot\frac{4}{5}\cdot\ldots\cdot\frac{2n}{2n+1} \le \int_0^{\sqrt{n}} e^{-y^2}\,dy \le \int_0^\infty e^{-y^2}\,dy \le \frac{\pi}{2}\sqrt{n}\frac{1}{2}\cdot\frac{3}{4}\cdot\ldots\cdot\frac{2n-3}{2n-2}.$$

(d) Now use Problem 40(d) to show that

$$\int_0^\infty e^{-y^2}\,dy = \frac{\sqrt{\pi}}{2}.$$

****42.** (a) Use integration by parts to show that

$$\int_a^b \frac{\sin x}{x}\,dx = \frac{\cos a}{a} - \frac{\cos b}{b} - \int_a^b \frac{\cos x}{x^2}\,dx,$$

and conclude that $\int_0^\infty (\sin x)/x\,dx$ exists. (Use the left side to investigate the limit as $a \to 0^+$ and the right side for the limit as $b \to \infty$.)

(b) Use Problem 15-33 to show that

$$\int_0^\pi \frac{\sin(n+\frac{1}{2})t}{\sin\dfrac{t}{2}}\,dt = \pi$$

for any natural number n.

(c) Prove that

$$\lim_{\lambda\to\pi}\int_0^\pi \sin(\lambda+\tfrac{1}{2})t\left[\frac{2}{t} - \frac{1}{\sin\dfrac{t}{2}}\right]dt = 0.$$

Hint: The term in brackets is bounded by Problem 15-2(vi); the Riemann-Lebesgue Lemma then applies.

(d) Use the substitution $u = (\lambda + \frac{1}{2})t$ and part (b) to show that

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

43. Given the value of $\int_0^\infty (\sin x)/x\,dx$ from Problem 42, compute

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx$$

by using integration by parts. (As in Problem 37, the formula for $\sin 2x$ will play an important role.)

***44.** (a) Use the substitution $u = t^x$ to show that

$$\Gamma(x) = \frac{1}{x}\int_0^\infty e^{-u^{1/x}}\,du.$$

(b) Find $\Gamma(\frac{1}{2})$.

***45.** (a) Suppose that $\dfrac{f(x)}{x}$ is integrable on every interval $[a, b]$ for $0 < a < b$, and that $\lim_{x\to 0} f(x) = A$ and $\lim_{x\to\infty} f(x) = B$. Prove that for all $\alpha, \beta > 0$ we have

$$\int_0^\infty \frac{f(\alpha x) - f(\beta x)}{x}\,dx = (A - B)\log\frac{\beta}{\alpha}.$$

Hint: To estimate $\displaystyle\int_\varepsilon^N \frac{f(\alpha x) - f(\beta x)}{x}\,dx$ use two different substitutions.

(b) Now suppose instead that $\displaystyle\int_a^\infty \frac{f(x)}{x}\,dx$ converges for all $a > 0$ and that $\lim_{x\to 0} f(x) = A$. Prove that

$$\int_0^\infty \frac{f(\alpha x) - f(\beta x)}{x}\,dx = A\log\frac{\beta}{\alpha}.$$

(c) Compute the following integrals:

(i) $\displaystyle\int_0^\infty \frac{e^{-\alpha x} - e^{-\beta x}}{x}\,dx.$

(ii) $\displaystyle\int_0^\infty \frac{\cos(\alpha x) - \cos(\beta x)}{x}\,dx.$

In Chapter 13 we said, rather blithely, that integrals may be computed to any degree of accuracy desired by calculating lower and upper sums. But an applied mathematician, who really has to do the calculation, rather than just talking about doing it, may not be overjoyed at the prospect of computing lower sums to evaluate an integral to three decimal places, say (a degree of accuracy that might easily be needed in certain circumstances). The next three problems show how more refined methods can make the calculations much more efficient.

We ought to mention at the outset that computing upper and lower sums might not even be practical, since it might not be possible to compute the quantities m_i and M_i for each interval $[t_{i-1}, t_i]$. It is far more reasonable simply to pick points x_i in $[t_{i-1}, t_i]$ and consider $\sum_{i=1}^{n} f(x_i)\cdot(t_i - t_{i-1})$. This represents the sum of the areas of certain rectangles which partially overlap the graph of f—see Figure 1 in the Appendix to Chapter 13. But we will get a much better result if we instead choose the trapezoids shown in Figure 2.

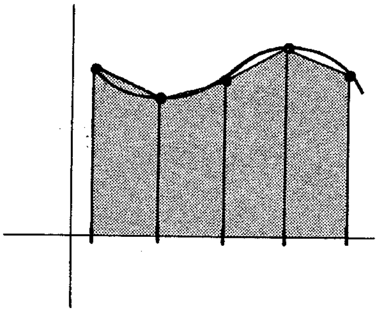


FIGURE 2

Suppose, in particular, that we divide $[a, b]$ into n equal intervals, by means of the points

$$t_i = a + i\left(\frac{b-a}{n}\right) = a + ih.$$

Then the trapezoid with base $[t_{i-1}, t_i]$ has area

$$\frac{f(t_{i-1}) + f(t_i)}{2}\cdot(t_i - t_{i-1})$$

and the sum of all these areas is simply

$$\begin{aligned}\Sigma_n &= h\left[\frac{f(t_1)+f(a)}{2} + \frac{f(t_2)+f(t_1)}{2} + \cdots + \frac{f(b)+f(t_{n-1})}{2}\right]\\ &= \frac{h}{2}\left[f(a) + 2\sum_{i=1}^{n-1} f(a+ih) + f(b)\right], \qquad h = \frac{b-a}{n}.\end{aligned}$$

This method of approximating an integral is called the *trapezoid rule.* Notice that to obtain Σ_{2n} from Σ_n it isn't necessary to recompute the old $f(t_i)$; their contribution to Σ_{2n} is just $\frac{1}{2}\Sigma_n$. So in practice it is best to compute Σ_2, Σ_4, Σ_8, ... to get approximations to $\int_a^b f$. In the next problem we will estimate $\int_a^b f - \Sigma_n$.

46. (a) Suppose that f'' is continuous. Let P_i be the linear function which agrees with f at t_{i-1} and t_i. Using Problem 11-43, show that if n_i and N_i are the minimum and maximum of f'' on $[t_{i-1}, t_i]$ and

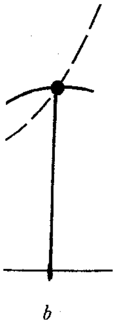
$$I = \int_{t_{i-1}}^{t_i} (x - t_{i-1})(x - t_i)\,dx$$

then

$$\frac{n_i I}{2} \geq \int_{t_{i-1}}^{t_i} (f - P_i) \geq \frac{N_i I}{2}.$$

(b) Evaluate I to get

$$\frac{n_i h^3}{12} \leq \int_{t_{i-1}}^{t_i} (f - P_i) \leq \frac{N_i h^3}{12}.$$



(c) Conclude that there is some c in $[a, b]$ with

$$\int_a^b f = \Sigma_n - \frac{(b-a)^3}{12n^2} f''(c).$$

Notice that the "error term" $(b-a)^3 f''(c)/12n^2$ varies as $1/n^2$ (while the error obtained using ordinary sums varies as $1/n$).

We can obtain still more accurate results if we approximate f by quadratic functions rather than by linear functions. We first consider what happens when the interval $[a, b]$ is divided into two equal intervals (Figure 3).

47. (a) Suppose first that $a = 0$ and $b = 2$. Let P be the second degree polynomial function which agrees with f at 0, 1, and 2 (Problem 3-6). Show that

$$\int_0^2 P = \frac{1}{3}[f(0) + 4f(1) + f(2)].$$

(b) Conclude that in the general case

$$\int_a^b P = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right].$$

(c) Naturally $\int_a^b P = \int_a^b f$ when f is a quadratic polynomial. But, remarkably enough, this same relation holds when f is a cubic polynomial! Prove this, using Problem 11-43; note that f''' is a constant.

The previous problem shows that we do not have to do any new calculations to compute $\int_a^b Q$ when Q is a *cubic* polynomial which agrees with f at a, b, and $\dfrac{a+b}{2}$: we still have

$$\int_a^b Q = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right].$$

But there is much more lee-way in choosing Q, which we can use to our advantage:

48. (a) Show that there is a cubic polynomial function Q satisfying

$$Q(a) = f(a), \qquad Q(b) = f(b), \qquad Q\left(\frac{a+b}{2}\right) = f\left(\frac{a+b}{2}\right)$$

$$Q'\left(\frac{a+b}{2}\right) = f'\left(\frac{a+b}{2}\right).$$

Hint: Clearly $Q(x) = P(x) + A(x-a)(x-b)\left(x - \frac{a+b}{2}\right)$ for some A.

(b) Prove that for every x we have

$$f(x) - Q(x) = (x-a)\left(x - \frac{a+b}{2}\right)^2 (x-b)\frac{f^{(4)}(\xi)}{4!}$$

for some ξ in $[a, b]$. Hint: Imitate the proof of Problem 11-43.

(c) Conclude that if $f^{(4)}$ is continuous, then

$$\int_a^b f = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{(b-a)^5}{2880} f^{(4)}(c)$$

for some c in $[a, b]$.

(d) Now divide $[a, b]$ into $2n$ intervals by means of the points

$$t_i = a + ih, \qquad h = \frac{b-a}{2n}.$$

Prove *Simpson's rule*:

$$\int_a^b f = \frac{b-a}{n}\left(f(a) + 4\sum_{i=1}^{n} f(t_{2i-1}) + 2\sum_{i=1}^{n-1} f(t_{2i}) + f(b)\right)$$

$$- \frac{(b-a)^5}{2880n^4} f^{(4)}(\bar{c})$$

for some $\bar{c}$ in $[a, b]$.