# Summary for Factoring

## 1 What you should know

The axiom of distribution is the basis of both expanding factors to produce a sum and of factoring a sum to a product. Although one can always use distribution directly, there are some forms that occur so frequently, that we work them without explicitly using distribution.

### 1.1 Products to sums

#### 1.1.1 Your best friends

The following are frequently encountered. You should be able to identify and use these forms nearly automatically.

1. Product of binomials. $(a+b)(c+d) = ab+bc+ad+bd$.
   Example:
   $$(x+2)(x+5) = x^2+2x+5x+10 = x^2+7x+10.$$
2. Sum squared. $(a+b)^2 = a^2+2ab+b^2$.
   Example:
   $$(x+3)^2 = x^2+6x+9.$$
3. Difference squared. $(a-b)^2 = a^2-2ab+b^2$.
   Example:
   $$(x-5)^2 = x^2-10x+25.$$
4. Sum times difference. $(a-b)(a+b) = a^2-b^2$.
   Example:
   $$(x-7)(x+7) = x^2-49.$$

#### 1.1.2 Acquaintances

Sometimes you will meet a product that appears unusual, but is in fact a cousin to one of the forms mentioned above. We are not expected to handle such products automatically. We may need to pause and think. But they should not require great effort or deep insight. A few examples follow.

**Example 1.** *Expand* $(x-2)(x+2)(x+4)(x+16)$.

**Solution.**

$(x-2)(x+2)(x+4)(x+16) = (x-4)(x+4)(x+16) = (x-16)(x+16) = x^2 - 256.$

**Example 2.** *Expand* $(a+b+3)^2$.

**Solution.**

Think of $a+b$ as a single object,

$$\begin{aligned}(a+b+3)^2 &= ((\mathbf{a}+\mathbf{b})+3)^2\\ &= (\mathbf{a}+\mathbf{b})^2 + 6(\mathbf{a}+\mathbf{b}) + 9.\end{aligned}$$

Now see the parts of $(a+b)$,

$$= a^2 + 2ab + b^2 + 6a + 6b + 9.$$

## 1.2 Sums to products

### 1.2.1 Special forms

You should be able to identify and use these forms nearly automatically.

1. Square of a sum. $a^2 + 2ab + b^2 = (a+b)^2$.
   Example:
   $$x^2 + 6x + 9 = (x+3)^2.$$
2. Square of a difference. $a^2 - 2ab + b^2 = (a-b)^2$.
   Example:
   $$x^2 - 10x + 25 = (x-5)^2.$$
3. Difference of squares. $a^2 - b^2 = (a-b)(a+b)$.
   Example:
   $$x^2 - 49 = (x-7)(x+7).$$

### 1.2.2 Factor by grouping

**Example 3.** *Factor* $4x^3 + 6x^2 + 6x + 9$.

**Solution.**

$$4x^3+6x^2+6x+9 = 2x^2(2x+3)+3(2x+3)$$

Notice that $(2x+3)$ is now a common factor of both terms of the sum. So,

$$= (2x+3)(2x^2+3). \quad \blacksquare$$

When we recall that expanding $(2x+3)(2x^2+3)$ amounts to distributing twice, we realize that factoring by grouping reverses the expansion, and it does so by using distribution twice.

### 1.2.3 Annoying forms

A trinomial that is not an instance of any special form of section (1.2.1) is annoying, unless one is fond of puzzles, because the method amounts to trial and error. If the leading coefficient and the constant each have few factors, you will find the factorization without too much effort. On the other hand, if either the leading coefficient or constant have many factors, the work can be tedious. Not harder. Just more trouble. There is but one path to acquiring the skill of factoring these types. That path is practice.

## 1.3 Advice

1. Always: if there is a common numerical factor, then begin by factoring that number out.
   Example:
   $$12x^2+24x-180.$$
   As written, this trinomial would be pure misery to factor, because the leading coefficient of 12 and the constant term of 180 have so many factors. But, factoring out 12 produces $12(x^2+2x-15)$ which is very easy to factor!

2. Always immediately check every factorization. Begin by checking the cross product, because that is where the mistake will show up.
   Example:
   $$x^2-5x-6 \stackrel{?}{=} (x-3)(x+2).$$
   But, the cross product of $(x-3)(x+2)$ is $-3x+2x = -x \neq -5x$.

3. Check to be sure an expression cannot be factored further.
   Example:
   $$x^4-16 = (x^2-4)(x^2+4)$$

But, $x^2-4$ factors into $(x-2)(x+2)$. So the complete factorization is $(x-2)(x+2)(x^2+4)$.

4. Look for an underlying friendly form.
   Example:

   $$(x+3)^2-25.$$

   See the difference of squares. So,

   $$\begin{aligned}(\mathbf{x}+\mathbf{3})^2-25 &= ((\mathbf{x}-\mathbf{3})-5)((\mathbf{x}-\mathbf{3})+5)\\ &= (x-8)(x+2).\end{aligned}$$

5. Do not overlook the fact that $1$ is a square number.
   Example:

   $$x^2-1$$

   is the difference of squares.

6. Nor should you forget that every number has factors of $1$ and itself.
   Example:

   $$x^2-5x-6=(x-6)(x+1).$$

7. Every factorization can be checked by expanding the factored form! You can know if your factorizations are correct before the exams are even collected.