1 The straight line

We have assumed that the equation $y = mx + b$ produces a set of points all of which lie on a straight line. There are all manner of lines that are not straight. For example, the graph of $y = x^5 - 2x^3$ shown in Figure (1) is a line, but not straight.

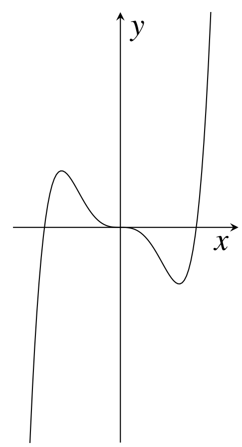
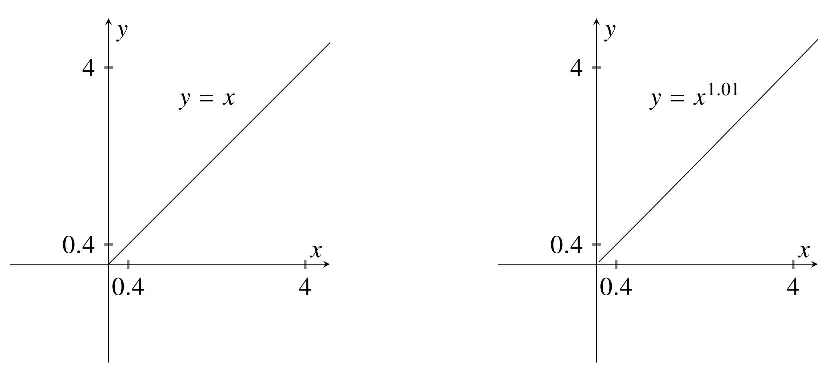


Figure 1: The graph of $y = x^5 - 2x^3$.

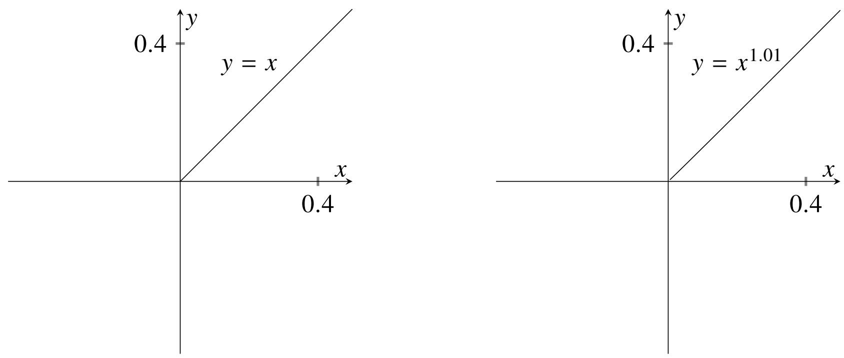
Figure (2) shows the accurately drawn graphs of two functions. One graph is a straight line and the other is not. Can you tell the difference?



(a) $y = x$. Straight line. (b) $y = x^{1.01}$. No straight line.

Figure 2: Both lines appear straight. But one is not.

Figure (3) provides a closer look at a small, $0 \leq x \leq 0.4$, portion of the graphs of Figure (2). Can you see the difference, now?



(a) $y = x$. Straight line. (b) $y = x^{1.01}$. No straight line.

Figure 3: Small portion of graphs of Figure (b) magnified 10 times.

Appearances can be deceiving. Both lines shown in Figure (3) look straight. The graphs provide no clue that one line is straight, the other not. Your mind and a little algebra can make clear that which your eyes cannot.

1.1 The graph of y = mx + b is a straight line.

We discovered that the essential quality of a straight line is that its direction does not change. We formalized this intuition by saying that a line whose slope is constant is a straight line.

The function $y = mx + b$, m and b constants, is called the **linear function**. Now we will show that the graph of a linear function is a straight line. Suppose that points $P(x_1, y_1), Q(x_2, y_2)$ are any two points on the graph of $y = mx + b$ and that $x_1 \neq x_2$ and x_1, x_2 are not both 0. Then,

$$\text{slope using points } P \text{ and } Q = \frac{y_2 - y_1}{x_2 - x_1}.$$

Since, $y_1 = mx_1 + b$ and $y_2 = mx_2 + b$, we substitute

$$\begin{aligned} \frac{(mx_2 + b) - (mx_1 + b)}{x_2 - x_1} &= \frac{m(x_2 - x_1)}{x_2 - x_1} \\ &= m. \end{aligned}$$

The slope using any two points is the same and a constant. This means the graph of the linear function is a straight line.

1.2 The equation of a straight line is y = mx + b.

Let $(x_1, y_1), (x_2, y_2)$, $x_1 \neq x_2$ and x_1, x_2 not both 0, be any two fixed points on the straight line ℓ and (x, y) be any other (variable) point on ℓ. Since ℓ is straight, the slope, m, computed using any pair of points on ℓ must equal the slope computed using any other pair of points on ℓ. This implies

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$m = \frac{y - y_1}{x - x_1}$$

so that

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \tag{1}$$

but $\dfrac{y_2 - y_1}{x_2 - x_1}$ is the slope, m, so

$$y - y_1 = m(x - x_1) \tag{2}$$

rearranging

$$\begin{aligned} y &= mx - mx_1 + y_1, \\ &= mx + (y_1 - mx_1), \end{aligned}$$

since m, x_1 and y_1 are constants, $y_1 - mx_1$ is a constant, call it b. Thus,

$$y = mx + b. \tag{3}$$

This shows that if the graph is a straight line, then the equation is $y = mx + b$.
■

Together Sections (1.1) and (1.2) show that the graph of every equation of the form $y = mx + b$ is a straight line and every non-vertical straight line has an equation of the form $y = mx + b$ where m is the slope of the line.

Theorem 1. *A non-vertical line in the xy-coordinate plane is a straight line if and only if it is the graph of $y = mx + b$, where m and b are constants and m is the slope of the line.*

Equation (1) is called the **two point form** of the equation of a line. This equation looks like it is useful when the equation of the line through two points is desired. In practice, most people use Equation (2) instead. See Example (2).

Equation (2) is called the **point-slope form** of the equation of a line. It is handy for finding the equation of a line when the line's slope and a point on the line are known or when two points on the line are known.

Equation (3) is called the **slope-intercept form** of the equation of a line. Its principal virtue is that one determines the slope and the y-intercept of the line by inspection.

Another form of the equation of a line that comes up often is

$$ax + by = c, \tag{4}$$

where neither a nor b are 0. Equation (4) is called the **standard form** equation of the line. When possible, we write Equation (4) with integer coefficients.

Yet another form of the equation for a line is Equation (5)

$$\frac{x}{a} + \frac{y}{b} = 1, \tag{5}$$

where neither a nor b are 0. This is called the **intercept form** of the equation of a line. The x-intercept is a and the y-intercept is b, each obtained by inspection.

The phrase "of the equation of a line" is awkward, so instead we will usually say "of a line". For example, instead of saying "Equation (2) is the point-slope form of the equation of a line," we will say "Equation (2) is the point-slope form of a line."

1.3 Applications

This section includes several examples of using the ideas we have discussed.

Example 1. Find the slope and y-intercept of $2x + 3y = 1$.

Solution. Rewrite the equation in slope-intercept form.

$$\begin{aligned} 2x + 3y &= 6 \\ 3y &= -2x + 6 \\ y &= \frac{-2}{3}x + 2. \end{aligned}$$

So the slope is $\frac{-2}{3}$ and the y-intercept is 2.

Example 2. Find the equation of the line through the points $(1,5),(4,9)$.

Solution. Use point-slope form of the line: $y - y_1 = m(x - x_1)$.

$$m = \frac{9-5}{4-1} = \frac{4}{3}.$$

Then

$$y - 5 = \frac{4}{3}(x - 1).$$

■

The computation of the slope m involves just a little mental arithmetic. So the point-slope equation, simpler than the two-point equation, is typically preferred.

Example 3. Find the equation of the line through the points $(-2,7),(3,5)$. Answer in standard form.

Solution. Use point-slope form of the line: $y - y_1 = m(x - x_1)$.

$$m = \frac{5-7}{3+2} = \frac{-2}{5}.$$

Then

$$\begin{aligned} y - 5 &= \frac{-2}{5}(x - 3) \\ 5y - 25 &= -2(x - 3) \\ 5y - 25 &= -2x + 6 \\ 2x + 5y &= 31. \end{aligned}$$

Example 4. Find the equation of the line through the points $(-2,-6),(-5,7)$. Answer in slope-intercept form.

Solution. Use point-slope form of the line: $y - y_1 = m(x - x_1)$.

$$m = \frac{7+6}{-5+2} = \frac{13}{-3} = \frac{-13}{3}.$$

Then

$$y + 2 = \frac{-13}{3}(x+6)$$

$$y + 2 = \frac{-13}{3}x - 26$$

$$y = \frac{-13}{3}x - 28$$

Example 5. Find the equation of the line through the points $(8,-3),(-1,9)$. Answer in standard form.

Solution. Use point-slope form of the line: $y - y_1 = m(x - x_1)$.

$$m = \frac{9+3}{-1-8} = \frac{12}{-9} = \frac{-4}{3}.$$

Then

$$y - 9 = \frac{-4}{3}(x+1)$$

$$3y - 27 = -4x - 4$$

$$4x + 3y = 23.$$

Example 6. Find the equation of the line through the points $\left(\frac{1}{2},3\right)\left(\frac{1}{3},\frac{3}{4}\right)$. Answer in standard form.

Solution. Use point-slope form of the line: $y - y_1 = m(x - x_1)$.

$$m = \frac{\frac{3}{4} - 3}{\frac{1}{3} - \frac{1}{2}}$$

$$= \frac{\frac{-9}{4}}{\frac{-1}{6}}$$

$$= \frac{27}{2}.$$

Then

$$\begin{aligned} y - 3 &= \frac{27}{2}\left(x - \frac{1}{2}\right) \\ 2y - 6 &= 27\left(x - \frac{1}{2}\right) \\ 2y - 6 &= 27x - \frac{27}{2} \\ 4y - 12 &= 54x - 27 \\ 54x - 4y &= 15. \end{aligned}$$

■

There is a method sometimes favored in schools, but otherwise ignored, that is used to find the equation of a line given two points on the line. Applied to Example (4), it would go like this. Since $(-2,-6),(-5,7)$ are points on the line, their coordinates satisfy the equation $y = mx + b$. Thus, a pair of simultaneous equations may be produced:

$$\begin{bmatrix} -2m + b = -6 \\ -5m + b = 7 \end{bmatrix}.$$

The system is solved for m and b resulting in $m = \frac{-13}{3}$ and $b = -28$. The equation is then written $y = -\frac{13}{3}x - 28$ as in Example (4).

Remark 1. *This method is usually ignored, because it is merely procedural. The letters m and b appear, but they need not be thought of a representing geometric features of a line. By contrast, the equation $y - y_1 = m(x - x_1)$ is natural when lines that pass through points and have slopes are the objects under discussion. Moreover, as a student using algebra to study the straight line, you are supposed to be thinking about slopes and points. Methods that reveal the nature of the straight line are preferable to methods that conceal that nature.*

Example 7. Find the equation of the line ℓ that is parallel to the line $y = \frac{3}{5}x - 4$ and passes through the point $(-5, 11)$. Answer in standard form.

Solution. Use point-slope form of the line: $y - y_1 = m(x - x_1)$. Since ℓ is parallel to $y = \frac{3}{5}x - 4$ it must have the same slope. That slope is, by inspection, $\frac{3}{5}$. Then,

$$\begin{aligned} y - 11 &= \tfrac{3}{5}(x+5) \\ 5y - 55 &= 3(x+5) \\ 5y - 55 &= 3x + 15 \\ 3x - 5y &= -70. \end{aligned}$$

Example 8. Find the x-intercept and the y-intercept of the line $ax + by = c$.

Solution. Rewrite the equation in intercept form $\dfrac{x}{a} + \dfrac{y}{b} = 1$.

$$\begin{aligned} &ax + by = c \\ &\frac{ax}{c} + \frac{by}{c} = 1 \\ &\frac{a}{c}x + \frac{b}{c}y = 1 \\ &\frac{x}{\frac{c}{a}} + \frac{y}{\frac{c}{b}} = 1. \qquad \text{WHY?} \end{aligned}$$

By inspection, the x-intercept is $\frac{c}{a}$ and the y-intercept is $\frac{c}{b}$.

Example 9. Find the x-intercept and the y-intercept of the line $3x + 2y = 5$.

Solution. Rewrite the equation in intercept form $\dfrac{x}{a} + \dfrac{y}{b} = 1$.

$$\begin{aligned} 3x + 2y &= 5 \\ \frac{3x}{5} + \frac{2y}{5} &= 1 \\ \frac{x}{\frac{5}{3}} + \frac{y}{\frac{5}{2}} &= 1. \end{aligned}$$

By inspection, the x-intercept is $\frac{5}{3}$ and the y-intercept is $\frac{5}{2}$. ■

The solution of Example (9) using the intercept-form is so slick, that anyone who is familiar with it would probably just eyeball $3x + 2y = 5$, then state the intercepts. But the criticism of Remark (1) page 7 might apply to it.

An excellent alternative is to think about what “x-intercept” and “y-intercept” mean. The line will cross the x-axis when $y = 0$ and it will cross the y-axis when $x = 0$.

Example 10. Find the x-intercept and the y-intercept of the line $4x+3y = 7$.

Solution.

When $y = 0$

$$\begin{aligned} 4x + 0 &= 7 \\ x &= \frac{7}{4}. \end{aligned}$$

When $x = 0$

$$\begin{aligned} 0 + 3y &= 7 \\ y &= \frac{7}{3}. \end{aligned}$$

So, the x-intercept is $\frac{7}{4}$ and the y-intercept is $\frac{7}{3}$.

Example 11. Find the slope and y-intercept of the line $ax + by = c$.

Solution. Rewrite the equation in slope-intercept form $y = mx + b$.

$$\begin{aligned} ax + by &= c \\ by &= -ax + c \\ y &= \frac{-a}{b}x + \frac{c}{b}. \end{aligned}$$

By inspection, the slope is $\frac{-a}{b}$ and the y-intercept is $\frac{c}{b}$.

Example 12. Find the point at which lines ℓ_1 and ℓ_2 intersect if the equation for line ℓ_1 is $2x + 3y = -6$ and the equation of line ℓ_2 is $x - 3y = 6$.

Solution. The point of intersection must be a point on each of the two lines. So, its coordinates must make each equation true. The coordinates are easily found by solving for x and y.

$$\begin{bmatrix} 2x + 3y = -6 \\ x - 3y = 6 \end{bmatrix}.$$

The point of intersection is found to be $(0, -2)$. Figure (4) illustrates this solution.

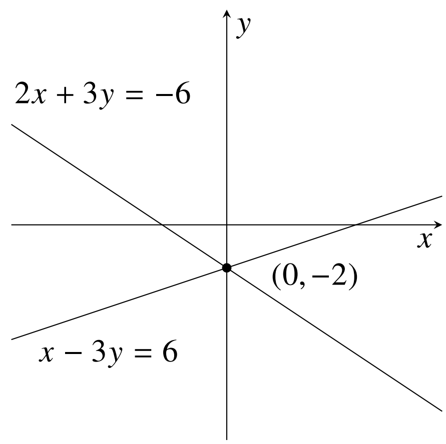


Figure 4: The point of intersection of two lines must be on each line.

Example 13. Find the point at which lines ℓ_1 and ℓ_2 intersect if the equation for line ℓ_1 is $y = \frac{2}{3}x + 2$ and the equation of line ℓ_2 is $y = 2x - 5$.

Solution. The point of intersection must be a point on each of the two lines. So, its coordinates must make each equation true. Since $\frac{2}{3}x + 2$ and $2x - 5$ both equal y,

$$\begin{aligned} \frac{2}{3}x + 2 &= 2x - 5 \\ 2x + 6 &= 6x - 15 \\ 4x &= 21 \\ x &= \frac{21}{4}. \end{aligned}$$

When $x = \frac{21}{4}, y = 2\left(\frac{21}{4}\right) - 5 = \frac{11}{2}$. So the point of intersection is $\left(\frac{21}{4}, \frac{11}{2}\right)$.

Example 14. Find the point at which lines ℓ_1 and ℓ_2 intersect if the equation for line ℓ_1 is $y = \frac{11}{19}x + 2$ and the equation of line ℓ_2 is $\frac{11}{19}x + 1$.

Solution. Look and think before you leap. The lines have the same slope. They are two different lines, because ℓ_1 goes through $(0, 2)$ but ℓ_2 goes through $(0, 1)$. The lines are parallel, so cannot intersect. Answer: The point of intersection does not exist. Figure (5).

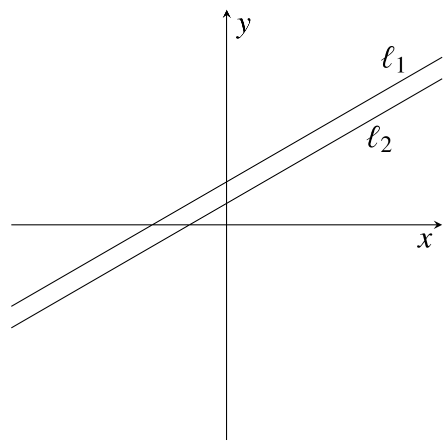


Figure 5: Parallel lines have no point in common.

Example 15. Find the point at which lines ℓ_1 and ℓ_2 intersect if the equation for line ℓ_1 is $2x + 3y = 18$ and the equation of line ℓ_2 is $4x + 6y = 36$.

Solution. Notice that there is only one equation, because $4x + 6y = 36 \iff 2x + 3y = 18$. Answer: Every point on line $2x + 3y = 18$ is a point of intersection. Figure (6).

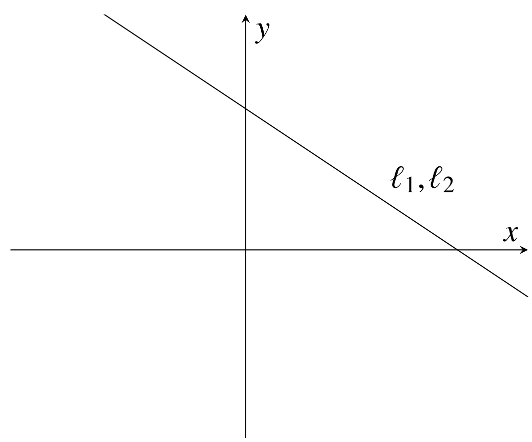


Figure 6: "ℓ_1", and "ℓ_2" name the same line.

1.4 Summary

Of the several forms of the equation for the straight line, the three listed below are the ones you must know and understand. You should know the following equations by their names.

- Point-slope: $y - y_1 = m(x - x_1)$.
- Slope-intercept: $y = mx + b$.
- Standard: $ax + by = c$.

You should know what the letters in these equations represent.

You should know that the point-slope equation is fundamental and that the other two important equations are rearrangements of the point-slope equation.

Ideally, you should be able to derive the point-slope equation of a line, given that the line is a straight line.

Exercise 1

1. The discussion of subsection (1.1), page 2, showed the graph of $y = mx + b$, m constant, is a straight line. Try to use a similar argument to show that graph of the function $y = mx^2 + b$, m constant, $m \neq 0$ is a straight line and note where the argument fails.
2. Continue the argument from question 1 to show that the graph of $y = mx^2 + b$ cannot be a straight line.
3. In question (2), you probably showed that the slope of $y = mx^2 + b$ is not constant, then concluded that the graph of $y = mx^2 + b$ cannot be a straight line. But, you would have begun by assuming that m is constant. How can it be that m is constant, but the slope of $y = mx^2 + b$ is *not* constant?
4. In question (1), why require that for $y = mx^2 + b$, $m \neq 0$?
5. On page 3, we replaced, $y_1 - mx_1$ with a single constant b. What justifies our doing that?
6. Is the sum of two linear functions a linear function? Give a reason for your answer.
7. Theorem (1) page 3 limits itself to non-vertical lines. Why?
8. Figure (b) on page 13 asserts that $y = x$ is a straight line but that $y = x^{1.01}$ is not a straight line.
   a) Prove that $y = x$ is a straight line.
   b) Prove that $y = x^{1.01}$ is not a straight line.

Answers to Exercise 1

(**1**) Let $y = mx^2 + b$ where m and b are constants. Suppose $P(x_1, y_1)$ and $Q(x_2, y_2)$ are any two points on the graph where $x_1 \neq x_2$ and x_1, x_2 are not both zero.

$$\text{slope using points } P \text{ and } Q = \frac{y_2 - y_1}{x_2 - x_1}.$$

Since, $y_1 = mx_1 + b$ and $y_2 = mx_2 + b$, we substitute

$$\frac{(mx_2^2 + b) - (mx_1^2 + b)}{x_2 - x_1} = m\frac{x_2^2 - x_1^2}{x_2 - x_1}.$$

$$\text{slope using points } P \text{ and } Q = m\frac{x_2^2 - x_1^2}{x_2 - x_1}.$$

The trouble is that although $m\dfrac{x_2^2 - x_1^2}{x_2 - x_1}$ is a constant, we cannot be certain of obtaining the same constant for every pair of points on the graph.

(**2**) To show that the graph of $y = mx^2 + b$ cannot be a straight line, we must go further than the answer to question (1). We must show that there is some pair of points for which the slope is not the same number. Let us select any third point $R(x_3, y_3)$ on the graph such that $x_3 \neq x_1$ and x_1 and x_3 are not both 0. Then,

$$\text{slope using points } P \text{ and } R = \frac{y_3 - y_1}{x_3 - x_1}.$$

Since, $y_1 = mx_1 + b$ and $y_3 = mx_3 + b$, we substitute

$$\frac{(mx_3^2 + b) - (mx_1^2 + b)}{x_3 - x_1} = m\frac{x_3^2 - x_1^2}{x_3 - x_1}.$$

$$\text{slope using points } P \text{ and } R = m\frac{x_3^2 - x_1^2}{x_3 - x_1}.$$

If there are three numbers in the domain of the function $y = mx^2 + b$ for which the slope using points P and $Q \neq$ slope using points P and R, the we are certain the graph is not a straight line. Choose $x_1 = 1, x_2 = 2$ and $x_3 = 3$. Then,

$$\text{slope using points } P \text{ and } R = m\frac{x_3^2 - x_1^2}{x_3 - x_1}$$
$$= m\frac{3^2 - 1^2}{3 - 1}$$
$$= \frac{8}{2}m.$$
$$= 4m.$$

But,

$$\text{slope using points } P \text{ and } Q = m\frac{x_2^2 - x_1^2}{x_2 - x_1}$$
$$= m\frac{2^2 - 1^2}{2 - 1}$$
$$= 3m.$$

Since the slopes using these two pairs of points are not equal, the slope is not constant. Therefore the line is not straight.

(**3**) m in $y = mx^2 + b$ is not slope.

(**4**) Because if $m = 0$, $y = b$. But, $y = b$ *is* the equation of a straight line through $(0, b)$ and parallel to the x-axis.

(**5**) Since the real numbers are closed under multiplication and subtraction, $y_1 - mx_1$ is a real number. We name it b.

(**6**) Yes. To see why, let $y = m_1x + b_1$ and $y = m_2x + b_2$ be two linear functions. Then the sum $(m_1x + b_1) + (m_2x + b_2) = (m_1 + m_2)x + (b_1 + b_2)$. Now $m_1 + m_2$ is a constant, call it m and $b_1 + b_2$ is a constant, call it b. But $y = mx + b$, m and b constant, is a linear function.

(**7**) The theorem limits itself to non-vertical lines because the slope of a vertical line would be $\frac{y_2 - y_1}{0}$ which is nonsense.

(**8**) (**a**) $y = x$ is the form $y = mx + b$ when $m = 1$ and $b = 0$. So by Theorem (1), its graph is a straight line.
(**b**) $y = x^{1.01}$ is not of the form $y = mx + b$ because the exponent on x is 1.01 not 1. Therefore, by Theorem (1), its graph is not a straight line.