Chapter 1

Integers

As you have seen, the integers provide the foundation for the other important kinds of numbers, such as the rational numbers. The study of the integers is an important area of mathematics called "Number Theory". Although the integers might seem simple compared to other kinds of numbers, they have furnished some of the most compelling problems in mathematics as well as some of the most difficult. The ideas and techniques that you will learn in this chapter are mainly those that will be of constant value to you throughout your study of mathematics.

From earlier grades, you are already familiar with the ideas of factor and multiple. Even so, it will not hurt to be sure we are all talking about the same ideas.

1.1. Factors, divisors, and multiples

If a number is a product of several positive integers, we call those several positive integers "factors" of the number. The number is a "multiple" of each of its factors.

Example 1.1

Find all the factors of 12.

Solution

Since $3 \cdot 4 = 12$, 3 and 4 are each factors of 12. Of course, 12 also equals $2 \cdot 6$, so 2 and 6 are also factors of 12. Let us not overlook $1 \cdot 12 = 12$, so that 1 and 12 get to be factors of 12, too. Unless we have missed some, the factors of 12 are $1, 2, 3, 4, 6, 12$. ■

We also know that:

12 is a multiple of 1, because $12 = 1 \cdot 12$,
12 is a multiple of 2, because $12 = 2 \cdot 6$,
12 is a multiple of 3, because $12 = 3 \cdot 4$,
12 is a multiple of 4, because $12 = 3 \cdot 4$,
12 is a multiple of 6, because $12 = 2 \cdot 6$,
12 is a multiple of 12, because $12 = 1 \cdot 12$.

Example 1.2

Find all the multiples of 2.

Solution

All the multiples are 2 are provided by $2n$ where n is a positive integer. As n ranges over the positive integers

$$1, 2, 3, \cdots,$$

$2n$ takes the values

$$2, 4, 6, \cdots.$$

Example 1.3

Find the first 6 multiples of 3.

Solution

The first 6 multiples of 3 are provided by $3n$ where $n = 1, 2, 3, 4, 5, 6$. They are $3, 6, 9, 12, 15, 18$. ■

When an number can be divided without remainder by a positive integer that we will call p, we say that p is a "divisor" of the number.

Example 1.4

6 is a divisor of 72, because $72 \div 6 = 12$ with no remainder. But, 6 is not a divisor of 75 because $75 \div 6$ leaves a remainder of 3. ■

We may as well state as definitions these ideas about what are factors, divisors, and multiples.

Definition 1.1 (Factor)
A positive integer p is a **factor** of a number N, if $N = p \cdot q$, where q is also a positive integer.

Definition 1.2 (Divisor)
A positive integer p is a **divisor** of a number N, if $N \div p$ has remainder 0. When p is a divisor of N, we say "p divides N" or "N is divisible by p".

Definition 1.3 (Multiple)
A number N is a **multiple** of a positive integer p, if $N = p \cdot q$, where q is also a positive integer.

If you are thinking that there is hardly any difference between a factor and a divisor, you are correct. We will use the terms "factor" and "divisor" interchangeably.

Exercise 1.1

1. List all of the factors of each of the following numbers.

   a) 6 d) 45
   b) 21 e) 36
   c) 12 f) 7

2. List the first 6 multiples of each of the following numbers.

   a) 5 d) 10
   b) 6 e) 7
   c) 1 f) 101

3. Is 7 a divisor of 42?
4. Is 8 a divisor of 63?
5. What is the largest number less than 1000 that is divisible by 6?
6. Prove that 7 is not a divisor of 23.
7. Of the multiples of 3 less than 30, how many are also multiples of 6?
8. What is the smallest number that is a multiple of 3 and 5?
9. What is the smallest number that is a multiple of 3 and 9?
10. What number is the greatest factor of both 12 and 28?

1.2. Prime numbers

Prime numbers have fascinated human beings since our ancestors first noticed them. The primes form no recognizable pattern. As hard as people have tried, no one has discovered a formula that produces all and only prime numbers. A theorem called the Fundamental Theorem of Arithmetic guarantees that every integer is either a prime number or a product of prime numbers. This means the prime numbers are the building blocks of the integers. In spite of this, it is very difficult to discover the prime factors of a very large non-prime number. This fact has a practical application. The cryptographic scheme that protects millions of electronic communications every day succeeds because of the extreme difficulty of finding the prime factors of very large numbers.

Definition 1.4 (Prime number)
A number is a **prime number** if it has exactly two factors, 1 and itself.

Definition 1.5 (Composite number, non-prime number)
A number that is not a prime number is called a **non-prime number** or a **composite number**.

1.2.1. Sieve of Eratosthenes

One method of finding the prime numbers up to a number N is called the Sieve of Eratosthenes. It works like this. List the numbers from 2 to N. Circle 2, the first prime number. Cross out all the multiples of 2. The first number after 2 that is not crossed out must be a prime number, indeed it is the number 3. Circle it. Cross out all the multiples of 3. The first number after 3 that is not crossed out must be a prime number. In fact it is 5. Circle it. Continuing this process until every number up to N has been either circled or crossed out identifies all the primes (they are circled) up to N.

Using the Sieve of Eratosthenes to find all the prime numbers less than 100 is left as an exercise. When you do it, you will discover that the numbers less than 100 which are prime are

$$2,3,5,7,11,13,17,19,23,29,31,37,41,$$
$$43,47,53,59,61,67,71,73,79,83,89,97.$$

Example 1.5

Is 27 a prime number?

Solution

No. 3 is a factor of 27, so 27 has a factor other than 1 and itself.

Example 1.6

Is 23 a prime number?

Solution

Yes. The only factors of 23 are 1 and 23.

1.3. Prime factorization

The primes are the building blocks of the integers. A systematic procedure to find the prime factorization of a number would be quite welcome. The examples that follow demonstrate such a procedure.

Example 1.7

Find the prime factorization of 12.

Solution

The first prime number is 2. Keep dividing by 2 until 3 is obtained which is not divisible by 2. The next prime number after 2 is 3. Divide by 3. The appearance of the number 1 terminates the process.

$$\begin{array}{r|l} 2 & 12 \\ \hline 2 & 6 \\ \hline 3 & 3 \\ \hline & 1 \end{array}$$

Therefore, $12 = 2 \cdot 2 \cdot 3$.

Example 1.8

Find the prime factorization of 50.

Solution

The first prime number is 2. Division by 2 produces 25 which is not divisible by 2. The next prime number after 2 is 3. But, 25 is not divisible by 3. The next prime number after 3 is 5. Keep dividing by 5. The appearance of the number 1 terminates the process.

$$\begin{array}{r|l} 2 & \underline{50} \\ 5 & \underline{25} \\ 5 & \underline{5} \\ & 1 \end{array}$$

Therefore, $50 = 2 \cdot 5 \cdot 5$.

Example 1.9

Find the prime factorization of 455.

Solution

The first prime number is 2. But, 2 is not a divisor of 255. The next prime number is 3. But, again, no divisor. The next prime number is 5 and it does divide 455.

$$\begin{array}{r|l} 5 & \underline{455} \\ 7 & \underline{91} \\ 13 & \underline{13} \\ & 1 \end{array}$$

Therefore, $455 = 5 \cdot 7 \cdot 13$.

Example 1.10

Completely factor 72.

Solution

$$\begin{array}{r|l} 2 & \underline{72} \\ 2 & \underline{36} \\ 2 & \underline{18} \\ 3 & \underline{9} \\ 3 & \underline{3} \\ & 1 \end{array}$$

Therefore, $72 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$. ■

The result $72 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$ can be written more compactly using exponents. We write three factors of 2 as 2^3. We write two factors of 3 as 3^2. So $72 = 2^3 3^2$.

Example 1.11

Write 504 as a product of prime factors.

Solution

$$\begin{array}{rl} 2 & \lfloor \underline{504} \\ 2 & \lfloor \underline{252} \\ 2 & \lfloor \underline{126} \\ 3 & \lfloor \underline{63} \\ 3 & \lfloor \underline{21} \\ 7 & \lfloor \underline{7} \\ & \;1 \end{array}$$

Therefore, $504 = 2^3 3^2 7$.

Exercise 1.2

1. Write each of the following using exponents.

   a) $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$
   b) $2 \cdot 2 \cdot 2 \cdot 5 \cdot 5$
   c) $7 \cdot 11 \cdot 13 \cdot 13 \cdot 19$
   d) $2 \cdot 3 \cdot 7 \cdot 7 \cdot 7 \cdot 7$
   e) $2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 5 \cdot 11 \cdot 11$

2. Write each of the following numbers as a product of prime factors.

   a) 330
   b) 28
   c) 88
   d) 35
   e) 325
   f) 90
   g) 4125
   h) 1728
   i) 2646
   j) 132

3. Find each product.

   a) $2^3 5$
   b) 3^3
   c) $2 \cdot 3^3$
   d) $2^2 3^2$
   e) 2^4
   f) $2^2 5^2$
   g) 10^2
   h) 10^3
   i) 10^4
   j) 5^3
   k) $2 \cdot 3 \cdot 10^2$
   l) $2^2 \cdot 3 \cdot 7^2$

1.4. Greatest common divisor

The number which is the greatest common divisor of several numbers, must meet two conditions.

(1) the number must be a divisor of each of the several numbers
(2) the number must be the greatest of those divisors.

The phrase “greatest common divisor” is abbreviated “GCD”. The phrase “greatest common factor” is synonymous to “greatest common divisor” . The abbreviation for “greatest common factor” is “GCF”.

Example 1.12

Find the greatest common divisor of 12 and 30.

Solution

The divisors of 12 are $1, 2, 3, 4, 6, 12$.
The divisors of 30 are $1, 2, 3, 5, 6, 10, 15, 30$.
The common divisors of 12 and 30 are $1, 2, 3, 6$. Of these common divisors, 6 is the greatest.
Therefore the greatest common divisor of 12 and 30 is 6. A compact way to write this is $\mathsf{GCD}[12, 30] = 6$.

Example 1.13

Find the greatest common divisor of 42 and 105.

Solution

The divisors of 42 are $1, 2, 3, 6, 7, 14, 21, 42$.
The divisors of 105 are $1, 3, 5, 7, 15, 21, 35, 105$.
The common divisors of 42 and 105 are $3, 7, 21$. Of these common divisors, 21 is the greatest.
Therefore $\mathsf{GCD}[42, 105] = 21$.

When the two numbers whose GCD is desired do not have many divisors, as in examples 1.12 and 1.13, listing the divisors and choosing the greatest of the common divisors is a practical strategy. But, what about when the number of divisors is large? Or when the GCD of many numbers is desired?

For example, suppose we wish to know $\mathsf{GCD}[420, 660]$? Just finding all the divisors or 420 and of 660 is annoying. If you are not sure how annoying, go ahead and do it.

Do not despair. A little thought can eliminate a lot of computation. Example (1.14) shows a less laborious approach.

Example 1.14

Find the greatest common divisor of 420 and 660.

Solution

The prime factorizations of these numbers are not hard to obtain.

$$420 = 2^2 \cdot 3 \cdot 5 \cdot 7$$

$$660 = 2^2 \cdot 3 \cdot 5 \cdot 11$$

Each number has two factors of 2, one factor of 3 and one factor of 5. So, the greatest common divisor is $4 \cdot 3 \cdot 5 = 60$.

Example 1.15

Find the greatest common divisor of 120 and 140.

Solution

The prime factorizations of these numbers are:

$$120 = 2^3 \cdot 3 \cdot 5$$

$$140 = 2^2 \cdot 5 \cdot 7$$

Each number has two factors of 2 and one factor of 5. So $\mathrm{GCD}[120, 140] = 20$. ■

Notice that 120 has 3 factors of 2, but 140 has only 2 factors of 2. So two, not three, factors of 2 appear in the greatest *common* factor of 120 and 140.

Suppose we have been provided the prime factorizations of two numbers, call them A and B, shown below. How do we determine which factors are present in the greatest common divisor, $\mathrm{GCD}[A, B]$?

Although we usually do not write an exponent of 1, we will now, because it helps make the method obvious.

$$A = 2^3 \cdot 3^5 \cdot 5^1 \cdot 7^2 \cdot 13^4$$

$$B = 2^3 \cdot 3^3 \cdot 5^6 \cdot 7^3 \cdot 13^3$$

Remember the exponent tells how many of a factor there are. All the factors common to both A and B are $2^3, 3^3, 5^1, 7^2, 13^3$. So

$$\mathrm{GCD}[A, B] = 2^3 \cdot 3^3 \cdot 5^1 \cdot 7^2 \cdot 13^3.$$

Example 1.16

Find the greatest common divisor of

$$A = 2^2 \cdot 3^4 \cdot 7^2 \cdot 11$$
$$B = 2^3 \cdot 3^3 \cdot 5^6 \cdot 7^3 \cdot 13.$$

Solution

Number A has no factor of 5 and no factor of 13, so 5 and 13 will not appear in $\mathrm{GCD}[A, B]$. Since B has no factor of 11, the number 11 is not a common factor. So,

$$\mathrm{GCD}[A, B] = 2^2 \cdot 3^3 \cdot 7^2.$$

If we define a number with an exponent of 0, we can write a simple rule for finding the greatest divisor.

Definition 1.6 (Zero exponent)
For any number x, $x^0 = 1$.

Rule: to find the greatest common divisor of several numbers, write the prime factorization of each using the exponent 0 for missing factors. Then choose each factor using the smallest exponent.

Example 1.17

Find the greatest common divisor of 540 and 350.

Solution

$$540 = 2^2 \cdot 3^3 \cdot 5^1 \cdot 7^0$$
$$350 = 2^1 \cdot 3^0 \cdot 5^2 \cdot 7^1.$$

$$\therefore \quad \mathrm{GCD}[540, 350] = 2^1 \cdot 3^0 \cdot 5^1 \cdot 7^0 = 2 \cdot 1 \cdot 5 \cdot 1 = 10.$$

Example 1.18

Find the greatest common divisor of $63, 84$, and 490.

Solution

$$\begin{aligned} 63 &= 2^0 \cdot 3^2 \cdot 5^0 \cdot 7^1 \\ 84 &= 2^2 \cdot 3^1 \cdot 5^0 \cdot 7^1 \\ 490 &= 2^1 \cdot 3^0 \cdot 5^1 \cdot 7^2. \end{aligned}$$

$\therefore \quad \mathsf{GCD}[63, 84, 490] = 2^0 \cdot 3^0 \cdot 5^0 \cdot 7^1 = 7.$

Example 1.19

Find the greatest common divisor of 14 and 15.

Solution

$$\begin{aligned} 14 &= 2^1 \cdot 3^0 \cdot 5^0 \cdot 7^1 \\ 15 &= 2^0 \cdot 3^1 \cdot 5^1 \cdot 7^0. \end{aligned}$$

$\therefore \quad \mathsf{GCD}[540, 350] = 2^0 \cdot 3^0 \cdot 5^0 \cdot 7^0 = 1.$

Well, you probably already knew $\mathsf{GCD}[14, 15] = 1$ before the work of Example (1.19). Numbers whose greatest common divisor is 1 are called "relatively prime" numbers.

Definition 1.7 (Relatively prime)
Numbers whose greatest common divisor is 1 are called **relatively prime** numbers.

Exercise 1.3

1. Find the greatest common divisor of each pair of numbers.

a) $96, 80$
b) $48, 72$
c) $28, 98$
d) $76, 57$
e) $84, 63$
f) $39, 52$
g) $70, 28$
h) $42, 78$
i) $56, 70$
j) $48, 32$
k) $84, 78$
l) $96, 72$

2. Find the greatest common divisor of each pair of numbers.

a) $84, 126$
b) $105, 70$
c) $128, 160$
d) $69, 34$
e) $192, 96$
f) $144, 156$
g) $105, 168$
h) $178, 80$
i) $58, 116$
j) $114, 152$
k) $55, 143$
l) $66, 198$

3. Find the greatest common divisor of set of numbers.

a) $21, 92, 65$
b) $78, 72, 66$
c) $54, 72, 90$
d) $96, 72, 60$
e) $56, 98, 42$
f) $51, 68, 85$
g) $32, 64, 48$
h) $55, 66, 77$
i) $56, 40, 80$
j) $69, 92, 46$
k) $34, 85, 51$
l) $48, 80, 64$

4. The GCD of two numbers is 4 and the sum of the two numbers is 68. Find the two numbers.
5. The GCD of two numbers is 6 and the sum of the two numbers is 18. Find the two numbers.

1.5. Least common multiple

You have been finding the least common multiple of sets of numbers for several years. Every time you wished to add factions whose denominators were not identical, you found a least common denominator. That denominator was the least common multiple of all the denominators. We denote the least common multiple of numbers A and B by writing "$\mathrm{LCM}[A, B]$".

Often it is practical, as in Example (1.20) to simply write or imagine the multiples of two numbers. The first multiple that is common is the least common multiple.

Example 1.20

Find the least common multiple of 6 and 15.

Solution

The first few multiples of 6 and 15 are

$$6, 12, 18, 24, 30, 36 \cdots$$
$$15, 30, 45 \cdots .$$

The first common multiple we meet is 30 and this is the least common multiple. $\mathrm{LCM}[6, 15] = 30$. ■

Not being content to leave well enough alone, we wonder what is $\mathrm{LCM}[36, 196]$. We list the multiples of each number.

For 36:
$36, 72, 108, 144, 180, 216, 252, 288, 324, 360, 396, 432, 468, 504, 540, 576, 612, 648, 684, 720, 756, 792, 828, 864, 900, 936, 972, 1008, 1044, 1080, 1116, 1152, 1188, 1224, 1260, 1296, 1332, 1368, 1404, 1440, 1476, 1512, 1548, 1584, 1620, 656, 1692, 1728, 1764, \cdots$

For 196:
$196, 392, 588, 784, 980, 1176, 1372, 1568, 1764, \cdots$

Is there less exhausting way to obtain $\mathrm{LCM}[36, 196] = 1764$?

Often, we get insight into a number by showing its building blocks. That is, by writing the prime factorization of the number. Doing so for 36 and 196,

$$36 = 2^2 \cdot 3^2$$
$$196 = 2^2 \cdot 7^2.$$

A multiple of a number must have all the factors of that number. Multiples of 36 must have at least two factors of 2 and at least two factors of 3. Every multiple of 196 must have at least two factors of 2 and at least two factors of 7. The number whose factors are

$$2^2 \cdot 3^2 \cdot 7^2$$

has just the needed factors and no more. It is the least common multiple of 36 and 196. Therefore $\mathrm{LCM}[36,196] = 2^2 \cdot 3^2 \cdot 7^2 = 4 \cdot 9 \cdot 49 = 1764$.

Example 1.21

Find the least common multiple of 363 and 99.

Solution

$$363 = 3^1 \cdot 11^2$$
$$99 = 3^2 \cdot 11^1.$$

Therefore, $\mathrm{LCM}[363,99] = 3^2 \cdot 11^2 = 9 \cdot 121 = 1089$.

Example 1.22

Find $\mathrm{LCM}[140,165]$.

Solution

$$140 = 2^2 \cdot 3^0 \cdot 5^1 \cdot 7^1 \cdot 11^0$$
$$165 = 2^0 \cdot 3^1 \cdot 5^1 \cdot 7^0 \cdot 11^1.$$

Therefore, $\mathrm{LCM}[140,165] = 2^2 \cdot 3^1 \cdot 5^1 \cdot 7^1 \cdot 11^1$. ■

We can write a simple rule for finding the least common multiple of several numbers. Write the prime factorizations of each number. Choose every number that appears in either factorization and use the greatest exponent taken by the number.

Exercise 1.4

1. Find the least common multiple of each pair of numbers.

a) $36, 45$	g) $40, 30$
b) $18, 48$	h) $20, 22$
c) $36, 24$	i) $21, 28$
d) $12, 42$	j) $45, 10$
e) $24, 40$	k) $12, 16$
f) $14, 21$	l) $48, 36$

2. Find the least common multiple of each set of numbers.

a) $30, 50, 20$	d) $22, 44, 33$
b) $35, 10, 45$	e) $22, 6, 18$
c) $48, 24, 36$	f) $12, 20, 16$

3. The LCM of of 30 and another number is 840. Find the other number.

4. The GCD of two numbers is 1 and the LCM of the two numbers is 143. Find the two numbers.

5. Suppose that a cat returns to the same place in a barn every 21 days and that a mouse returns to that spot every 6 days. If the cat just met the mouse, but failed to catch the mouse, how many days later will the cat get her next chance to catch the mouse?

Answers to Exercise 1.1

(**1**) (**a**) $1,2,3,6$ (**b**) $1,3,7,21$ (**c**) $1,2,3,4,6,12$ (**d**) $1,3,5,9,15,45$
(**e**) $1,2,3,4,6,9,12,18,36$ (**f**) $1,7$
(**2**) (**a**) $5,10,15,20,25,30$ (**b**) $6,12,18,24,30,36$ (**c**) $1,2,3,4,5,6$
(**d**) $10,20,30,40,50,60$ (**e**) $7,14,21,28,35,42$ (**f**) $101,202,303,404,505,606$
(**3**) Yes. (**4**) No. (**5**) 996 (**6**) $23 \div 7 = 3$ with remainder $2 \neq 0$.
(**7**) 4 (**8**) 15 (**9**) 9 (**10**) 4

Answers to Exercise 1.2

(**1**) (**a**) 2^5 (**b**) $2^3\,5^2$ (**c**) $7 \cdot 11 \cdot 13^2\,19$ (**d**) $2 \cdot 3c\dot{7}^4$ (**e**) $2^2 3^2 5^3 11^2$
(**2**) (**a**) $2 \cdot 3 \cdot 5 \cdot 11$ (**b**) $2^2\,7$ (**c**) $2^3\,11$ (**d**) $5 \cdot 7$ (**e**) $5^2\,13$ (**f**) $2 \cdot 3^2\,5$
(**g**) $3 \cdot 5^3\,11$ (**h**) $2^6 3^2$ (**i**) $2 \cdot 3^3\,7^2$ (**j**) $2^2\,2 \cdot 11$
(**3**) (**a**) 40 (**b**) 27 (**c**) 54 (**d**) 36 (**e**) 16 (**f**) 100 (**g**) 100
(**h**) 1000 (**i**) 10000 (**j**) 125 (**k**) 600 (**l**) 588

Answers to Exercise 1.3

(**1**) (**a**) 16 (**b**) 24 (**c**) 14 (**d**) 19 (**e**) 21 (**f**) 13 (**g**) 14 (**h**) 6
(**i**) 14 (**j**) 16 (**k**) 6 (**l**) 24
(**2**) (**a**) 42 (**b**) 35 (**c**) 32 (**d**) 1 (**e**) 96 (**f**) 12 (**g**) 21 (**h**) 2
(**i**) 58 (**j**) 38 (**k**) 11 (**l**) 66
(**3**) (**a**) 1 (**b**) 6 (**c**) 18 (**d**) 12 (**e**) 14 (**f**) 17 (**g**) 16 (**h**) 11
(**i**) 8 (**j**) 23 (**k**) 17 (**l**) 16 (**4**) $8,60$ (**5**) $12,6$

Answers to Exercise 1.4

(**1**) (**a**) 180 (**b**) 144 (**c**) 72 (**d**) 84 (**e**) 120 (**f**) 42 (**g**) 120
(**h**) 220 (**i**) 84 (**j**) 90 (**k**) 48 (**l**) 144
(**2**) (**a**) 300 (**b**) 630 (**c**) 144 (**d**) 132 (**e**) 198 (**f**) 240
(**3**) 168 (**4**) 11 and 13 or 1 and 143 (**5**) 42 days