[12-12-03-T11]

Law of Sines (REV)

Much ink has been spilled over the "ambiguous case" in the context of using the law of sines. This is too bad, because the matter is trivial.

Suppose two sides and angle not included are given. As you know from geometry, SSA does not necessarily specify a unique triangle. Given angle α, side c, side a, two triangles are in general possible. Figure 1 shows the general case.

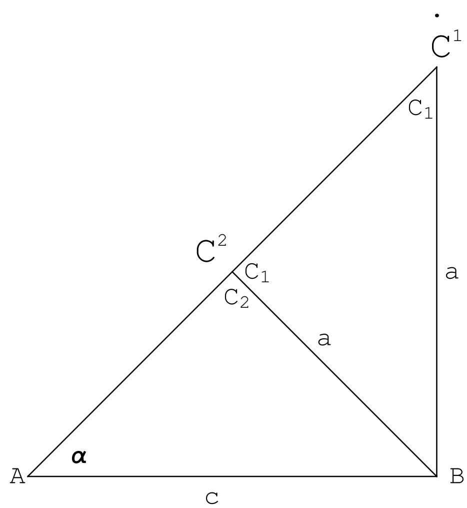

Figure 1

The question is, which of $\triangle ABC^1$, $\triangle ABC^2$ exist? The possible answers are "Both", "Neither", "$\triangle ABC^1$", or "$\triangle ABC^2$". It is easy to determine the answer.

Note that $\angle c_2 + \angle c_1 = 180°$. This means as soon one of $\angle c_1$ or $\angle c_2$ is known, so is the other. Furthermore,

$$\text{If } \alpha + c_1 \geq 180° \text{ then } \triangle ABC^1 \text{ does not exist.}$$

$$\text{If } \alpha + c_2 \geq 180° \text{ then } \triangle ABC^2 \text{ does not exist.}$$

(over)

EXAMPLE 1. Find angle β, if $\alpha = 47\,°$, $a = 25$ ft, $b = 30$ ft.

SOLUTION.

By the law of sines, $\frac{\sin 47\,°}{25\text{ ft}} = \frac{\sin\beta}{30\text{ ft}}$. Since, $\sin\beta = \frac{30\sin 47\,°}{25} \approx 0.8776$, $\beta \approx 61.35\,°$.

Check if such a triangle exists. $\alpha + \beta = 47\,° + 61.35\,° = 108.35\,° < 180\,°$. So, $\beta \approx 61.35\,°$ is a solution.

Referring to Figure 1, we say we have found c_1, or c_2, if you wish. The other possible solution, we'll call it β', is $\beta' = 180 - \beta = 180 - 61.35\,° = 118.65\,°$.

Check if such a triangle exists. $\alpha + \beta' = 47\,° + 118.65\,° = 165.65\,° < 180\,°$. So, $\beta' \approx 118.65\,°$ is a solution, too.

Therefore, $\beta \approx 61.35\,° \vee \beta \approx 118.65\,°$. ■

Note that if you had been asked to "solve the triangle", you would have to give the angles and lengths of sides of each of the triangles ABC^1 and ABC^2.

EXAMPLE 2. Find angle β, if $\alpha = 85\,°$, $a = 31$ ft, $b = 52$ ft.

SOLUTION.

By the law of sines, $\frac{\sin 85\,°}{31\text{ ft}} = \frac{\sin\beta}{52\text{ ft}}$. Since, $\sin\beta = \frac{52\sin 85\,°}{31} \approx 1.6710$, no triangle exists.

Therefore, there is no solution. ■

EXAMPLE 3. Find angle β, if $\alpha = 30\,°$, $a = 50\sqrt{3}$ ft, $b = 200$ ft.

SOLUTION.

By the law of sines, $\frac{\sin 30\,°}{100\text{ ft}} = \frac{\sin\beta}{200\text{ ft}}$. Since, $\sin\beta = \frac{200\sin 30\,°}{100} = 1$, the triangle is a right triangle.

Therefore, $\beta = 90\,°$. ■