1.1 Trigonometric Equations

1.1.1 Preliminary observations

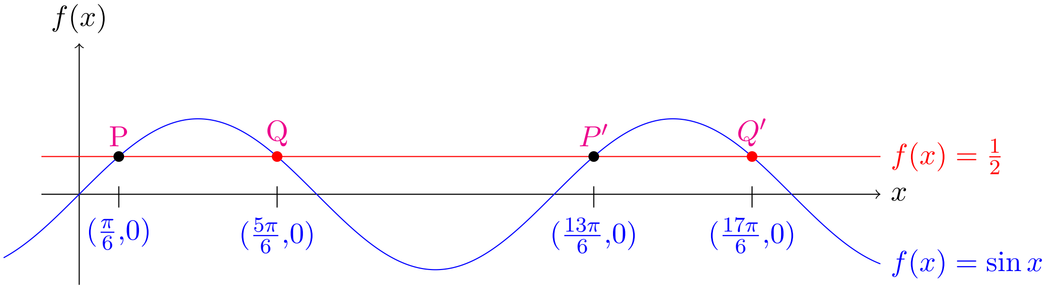

Figure 1.1: The sine function is not a 1-1 function on $\mathbb{R}$.

No trigonometric function is 1-1 on $\mathbb{R}$. Even the simple equation $\sin(x) = \dfrac{1}{2}$ has multiple solutions. In fact, there are two sequences of solutions. One sequence is $\{\frac{\pi}{6} + 2n\pi, n \in \mathbb{Z}\}$. The other is $\{\frac{5\pi}{6} + 2n\pi, n \in \mathbb{Z}\}$. It is no surprise that adjacent points in each sequence are 2π apart, since the period of $f(x) = \sin(x)$ is 2π.

Example 1.1. Solve $\sin(x) = \frac{1}{2}$, $x \in \mathbb{R}$.
Solution.
One sequence of solutions is based on $\sin(x + 2n\pi) = \dfrac{1}{2}$.

$$\begin{aligned} \sin(x + 2n\pi) = \frac{1}{2} &\implies x + 2n\pi = \frac{\pi}{6} \\ &\implies x = \frac{\pi}{6} + 2n\pi. \end{aligned} \tag{1.1}$$

And the other sequence of solutions is based on $\sin(\pi - x + 2n\pi) = \dfrac{1}{2}$.

$$\begin{aligned} \sin(\pi - x + 2n\pi) = \frac{1}{2} &\implies \pi - x + 2n\pi = \frac{\pi}{6} \\ &\implies -x = \frac{\pi}{6} - \pi + 2n\pi \\ &\implies x = \frac{5\pi}{6} + 2n\pi. \end{aligned} \tag{1.2}$$

Therefore,

$$x = \frac{\pi}{6} + 2n\pi \quad \text{or} \quad x = \frac{5\pi}{6} + 2n\pi,\ n \in \mathbb{Z}.$$

□

Are equations (1.1) and (1.2) correct? Should they have included "$-2n\pi$" instead of "$+2n\pi$"?

Let's agree to call the solutions that lie within one period of the function to the right of 0 the **principal solutions** of the equation. The principal solutions of $\sin(x) = \frac{1}{2}$ are $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$. Do note that the period is not always 2π. For example, the period of $f(x) = \sin(3x)$ is $\frac{2\pi}{3}$.

1.1.2 Solving equations

Several strategies for solving trigonometric equations are shown in the following examples. With practice, you will gain skill at solving equations that involve trigonometric functions. There are, however, equations that require mathematics beyond what you are learning here; for example, the equation $x \sin x = -1$.

Example 1.2. Solve $2\sin(x) + \sqrt{3} = 0,\ x \in \mathbb{R}$.
Solution.
First, rewrite $2\sin(x) + \sqrt{3} = 0$,

$$\begin{aligned} 2\sin(x) + \sqrt{3} = 0 &\iff \sin(x) = \frac{-\sqrt{3}}{2} \\ &\iff -\sin(x) = \frac{\sqrt{3}}{2}. \end{aligned}$$

Second, remember that $-\sin(\theta) = \sin(-\theta) = \sin(\pi + \theta)$.

One sequence of solutions is based on $\sin(-x + 2n\pi) = \dfrac{\sqrt{3}}{2}$.

$$\begin{aligned} \sin(-x + 2n\pi) = \frac{\sqrt{3}}{2} &\implies -x + 2n\pi = \frac{\pi}{3} \\ &\iff x = -\frac{\pi}{3} + 2n\pi \\ &\iff x = \frac{5\pi}{3} + 2n\pi. \end{aligned}$$

And the other sequence of solutions is based on $\sin(\pi + x + 2n\pi) = \dfrac{\sqrt{3}}{2}$.

$$\begin{aligned} \sin(\pi + x + 2n\pi) = \frac{\sqrt{3}}{2} &\implies \pi + x + 2n\pi = \frac{\pi}{3} \\ &\iff x = \frac{\pi}{3} - \pi + 2n\pi \\ &\iff x = \frac{-2\pi}{3} + 2n\pi \\ &\iff x = \frac{4\pi}{3} + 2n\pi. \end{aligned}$$

Therefore,

$$x = \frac{4\pi}{3} + 2n\pi \quad \text{or} \quad x = \frac{5\pi}{3} + 2n\pi,\ n \in \mathbb{Z}. \qquad \square$$

It may be illuminating to view graphs of the functions considered in this handout. Since the purpose would be to see the graphs, rather than learn how to produce them (you already learned that), you might as well use a machine to produce the graphs.

Example 1.3. Solve $\sin^2(x) + 7\sin(x) + 6 = 0,\ x \in \mathbb{R}$.

Solution.

$$\begin{aligned} \sin^2(x) + 7\sin(x) + 6 = 0 &\iff (\sin x + 1)(\sin x + 6) = 0 \\ &\iff \sin x + 1 = 0 \quad \text{or} \quad \sin x + 6 = 0 \\ &\iff \sin x = -1 \quad \text{or} \quad \sin x = -6 \\ &\implies x + 2n\pi = \frac{3\pi}{2} \quad \text{or} \quad x = \emptyset. \end{aligned}$$

Therefore,

$$x = \frac{3\pi}{2} + 2n\pi,\ n \in \mathbb{Z}. \qquad \square$$

Example 1.4. Solve $2\sin 4x = 1,\ x \in \mathbb{R}$.

Solution.

Rewrite $2\sin 4x = 1$ as $\sin 4x = \dfrac{1}{2}$.

One sequence of solutions is based on $\sin(4x + 2n\pi) = \dfrac{1}{2}$.

$$\begin{aligned} \sin(4x + 2n\pi) = \frac{1}{2} &\implies 4x + 2n\pi = \frac{\pi}{6} \\ &\iff 4x = \frac{\pi}{6} + 2n\pi \\ &\iff x = \frac{\pi}{24} + \frac{n\pi}{2}. \end{aligned}$$

And the other sequence of solutions is based on $\sin(\pi - 4x + 2n\pi) = \dfrac{1}{2}$.

$$\begin{aligned} \sin(\pi - 4x + 2n\pi) = \frac{1}{2} &\implies \pi - 4x + 2n\pi = \frac{\pi}{6} \\ &\iff -4x = \frac{\pi}{6} - \pi + 2n\pi \\ &\iff 4x = \pi - \frac{\pi}{6} + 2n\pi \\ &\iff 4x = \frac{5\pi}{6} + 2n\pi \\ &\iff x = \frac{5\pi}{24} + \frac{n\pi}{2}. \end{aligned}$$

Therefore,

$$x = \frac{\pi}{24} + \frac{n\pi}{2} \quad \text{or} \quad x = \frac{5\pi}{24} + \frac{n\pi}{2},\ n \in \mathbb{Z}. \qquad \square$$

Example 1.5. Solve $\sec^2 x + \tan^2 x = 3,\ x \in \mathbb{R}$.

Solution.

Rewrite $\sec^2 x + \tan^2 x = 3$ as $\sec^2 x + (\sec^2 x - 1) = 3$.

$$\begin{aligned}
\sec^2 x + (\sec^2 x - 1) = 3 &\iff 2\sec^2 x - 1 = 3 \\
&\iff 2\sec^2 x - 4 = 0 \\
&\iff \sec^2 x - 2 = 0 \\
&\iff (\sec x + \sqrt{2})(\sec x - \sqrt{2}) = 0 \\
&\iff \sec x = -\sqrt{2} \quad \text{or} \quad \sec x = \sqrt{2}. \qquad (1.3)
\end{aligned}$$

Each of equations (1.3) produces two sequences of solutions.

Two sequences of solutions from $\sec x = \sqrt{2}$ are

$$\sec x = \sqrt{2} \implies x + 2n\pi = \frac{\pi}{4} \iff x = \frac{\pi}{4} + 2n\pi$$

and

$$\sec x = \sqrt{2} \implies \sec(-x) = \sqrt{2} \implies -x + 2n\pi = \frac{\pi}{4} \iff x = \frac{-\pi}{4} + 2n\pi = \frac{7\pi}{4} + 2n\pi.$$

Two sequences of solutions from $\sec x = -\sqrt{2}$ are

$$\sec x = -\sqrt{2} \implies \sec(\pi - x) = \sqrt{2} \implies \pi - x + 2n\pi = \frac{\pi}{4} \iff x = \frac{3\pi}{4} + 2n\pi$$

and

$$\sec x = -\sqrt{2} \implies \sec(\pi + x) = \sqrt{2} \implies \pi + x + 2n\pi = \frac{\pi}{4} \iff x = \frac{-3\pi}{4} + 2n\pi = \frac{5\pi}{4} + 2n\pi.$$

So,

$$x = \frac{\pi}{4} + 2n\pi \vee x = \frac{3\pi}{4} + 2n\pi \vee x = \frac{4\pi}{4} + 2n\pi \vee x = \frac{7\pi}{4} + 2n\pi,\ n \in \mathbb{Z}.$$

You should convince yourself that all four of these sequences may be produced from

$$x = \frac{\pi}{4} + \frac{n\pi}{2},\ n \in \mathbb{Z}. \qquad \square$$

Example 1.6. Solve $\sin x - \cos x = 1,\ x \in \mathbb{R}$.

Solution.

The trouble is two functions. If we rewrite $\sin x - \cos x = 1$ as $\sin x = 1 + \cos x$, then square both sides, we can replace $\sin^2 x$ with $1 - \cos^2 x$.

$$\begin{aligned}\sin x - \cos x = 1 &\iff \sin x = 1 + \cos x\\ &\implies \sin^2 x = 1 + 2\cos x + \cos^2 x\\ &\iff 1 - \cos^2 x = 1 + 2\cos x + \cos^2 x\\ &\iff 2\cos x + 2\cos^2 x = 0\\ &\iff \cos x + \cos^2 x = 0\\ &\iff \cos x(1 - \cos x) = 0\\ &\implies \cos x = 0 \quad \text{or} \quad \cos x = 1.\end{aligned}$$

So *possible* solutions are

$$x = \frac{\pi}{2} + 2n\pi, \quad x = \frac{3\pi}{2} + 2n\pi, \quad x = \pi + 2n\pi.$$

But some of these may be extraneous having been introduced when we squared both sides of $\sin x = 1 + \cos x$. Checking,

$$\begin{aligned}x = \frac{\pi}{2} &\implies \sin x = 1, \cos x = 0, \sin x - \cos x = 1 - 0 = 1, \quad \text{(OK)}\\ x = \frac{3\pi}{2} &\implies \sin x = -1, \cos x = 0, \sin x - \cos x = -1 - 0 = -1, \quad \text{(NO)}\\ x = \pi &\implies \sin x = 0, \cos x = -1, \sin x - \cos x = 0 - (-1) = 1. \quad \text{(OK)}\end{aligned}$$

Therefore,

$$x = \frac{\pi}{2} + 2n\pi \quad \text{or} \quad x = \pi + 2n\pi,\ n \in \mathbb{Z}. \qquad \square$$

Example 1.7. Solve $\sin 3x = \sin 7x,\ x \in \mathbb{R}$.

Solution.

Remembering that $\sin\theta = \sin(\pi - \theta)$, we realize $\sin 3x = \sin 7x$ whenever $7x = 3x + 2n\pi$ or $7x = \pi - 3x + 2n\pi$. Then

$$\begin{aligned}7x = 3x + 2n\pi &\quad \text{or} \quad 7x = \pi - 3x + 2n\pi\\ 4x = 2n\pi &\quad \text{or} \quad 10x = \pi + 2n\pi.\end{aligned}$$

Therefore,

$$x = \frac{n\pi}{2} \quad \text{or} \quad x = \frac{\pi}{10} + \frac{n\pi}{5}. \qquad \square$$

Example 1.8. Solve $\cos 5x = -\cos 2x,\ x \in \mathbb{R}$.

Solution.

We use $-\cos\theta = \cos(\pi - \theta)$ and $-\cos\theta = \cos(\pi + \theta)$. So,

$$2x = \pi - 5x + 2n\pi \quad \text{or} \quad 2x = \pi + 5x + 2n\pi$$

$$7x = \pi + 2n\pi \quad \text{or} \quad -3x = \pi + 2n\pi.$$

Therefore,

$$x = \frac{\pi}{7} + \frac{2n\pi}{7} \quad \text{or} \quad x = \frac{-\pi}{3} + \frac{2n\pi}{3}.$$

Since $\cos(-\theta) = \cos\theta$, we could write

$$x = \frac{\pi}{7} + \frac{2n\pi}{7} \quad \text{or} \quad x = \frac{\pi}{3} + \frac{2n\pi}{3}. \qquad \square$$

Example 1.9. Solve $\sin 2x = \cos 3x,\ x \in \mathbb{R}$.

Solution.

Use $\cos\theta$ as $\sin(\frac{\pi}{2} - \theta)$. Then,

One sequence of solutions is based on $\sin 2x = \sin(\frac{\pi}{2} - 3x + 2n\pi)$.

$$\begin{aligned}\sin 2x = \sin\left(\frac{\pi}{2} - 3x + 2n\pi\right) &\implies 2x = \frac{\pi}{2} - 3x + 2n\pi \\ &\iff 5x = \frac{\pi}{2} + 2n\pi \\ &\iff x = \frac{\pi}{10} + \frac{2n\pi}{5}.\end{aligned}$$

And the other sequence of solutions is based on $\sin 2x = \sin(\pi - (\frac{\pi}{2} - 3x) + 2n\pi)$.

$$\begin{aligned}\sin 2x = \sin\left(\pi - \left(\frac{\pi}{2} - 3x\right) + 2n\pi\right) &\implies 2x = \pi - \left(\frac{\pi}{2} - 3x\right) + 2n\pi \\ &\iff 2x = \pi - \frac{\pi}{2} + 3x + 2n\pi \\ &\iff -x = \frac{\pi}{2} + 2n\pi \\ &\iff x = \frac{-\pi}{2} + 2n\pi.\end{aligned}$$

Therefore,

$$x = \frac{\pi}{10} + \frac{2n\pi}{5} \quad \text{or} \quad x = \frac{-\pi}{2} + 2n\pi,\ n \in \mathbb{Z}. \qquad \square$$

So far, all of the solutions have been exact. When an exact solution does not exist, we will write an approximate solution accurate to 5 significant figure.

Example 1.10. Solve $3\cos\frac{x}{3}+2=0,\ x\in\mathbb{R}$.

Solution.

Note that

$$\cos\frac{x}{3}=\frac{-2}{3} \iff -\cos\frac{x}{3}=\frac{2}{3} \implies \cos\left(\pi-\frac{x}{3}\right)=\frac{2}{3} \quad \text{and} \quad \cos\left(\pi+\frac{x}{3}\right)=\frac{2}{3}.$$

One sequence of solutions is based on $\cos\left(\pi-\frac{x}{3}\right)=\frac{2}{3}$.

$$\begin{aligned}\cos\left(\pi-\frac{x}{3}\right)=\frac{2}{3} &\implies \pi-\frac{x}{3}=0.84107+2n\pi \qquad (1.4)\\ &\iff -\frac{x}{3}=0.84107-\pi+2n\pi\\ &\iff x=3\pi-2.5232+6n\pi.\end{aligned}$$

And the other sequence of solutions is based on $\cos\left(\pi+\frac{x}{3}\right)=\frac{2}{3}$.

$$\begin{aligned}\cos\left(\pi+\frac{x}{3}\right)=\frac{2}{3} &\implies \pi+\frac{x}{3}=0.84107+2n\pi\\ &\iff \frac{x}{3}=0.84107-\pi+2n\pi\\ &\iff x=2.5232-3\pi+6n\pi.\end{aligned}$$

Therefore,

$$x=3\pi-2.5232+6n\pi \quad \text{or} \quad x=2.5232-3\pi+6n\pi,\ n\in\mathbb{Z}. \qquad \square$$

In equation (1.4), the value 0.84107 is obtained from a calculator as the inverse cosine of $\frac{2}{3}$.

Example 1.11. Suppose $\sin 2x = \frac{\sqrt{3}}{2}$. Find (a) the principal solutions, and (b) all the solutions in the interval $[0, 2\pi]$.

(a) Solution.

The period of $\sin 2x$ is π. The two sequences of solutions are

$$\frac{\pi}{5} + n\pi,\ n \in \mathbb{Z}$$

and

$$\frac{\pi}{3} + n\pi,\ n \in \mathbb{Z}.$$

The first few terms of the sequence $\left\{\frac{\pi}{6} + n\pi\right\}$ are $\left\{\frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}\right\}$ when $n = 0, 1, 2$. Only $\frac{\pi}{6}$ is in $[0, \pi]$.

The first few terms of the sequence $\left\{\frac{\pi}{3} + n\pi\right\}$ are $\left\{\frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}\right\}$ when $n = 0, 1, 2$. Only $\frac{\pi}{3}$ lies in $[0, \pi]$.

So, the only principal solutions are $\frac{\pi}{6}$ and $\frac{\pi}{3}$.

(b) Solution.

Since the frequency of $\sin 2x$ is 2, we expect that each sequence of solutions will produce two solutions in $[0, 2\pi]$. A glance at the solution to (a) above convinces us that this is indeed the case. The four solutions in $[0, 2\pi]$ are

$$\frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}. \qquad \square$$

1.2 Inverse functions

1.3 The sum sine and cosine

In general, the sum of the sine and cosine functions is difficult to compute. But when the frequency of the functions match, the sum is not too hard to find. Though a little ingenuity is required. Let's see if we can find the following sum.

$$y = a\sin kx + b\cos kx. \tag{1.5}$$

We multiply by 1 in a form that will initially seem arbitrary, but really isn't.

$$\begin{aligned} a\sin kx + b\cos kx &= \frac{\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}}\left(a\sin kx + b\cos kx\right) \\ &= \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\sin kx + \frac{b}{\sqrt{a^2+b^2}}\cos kx\right). \end{aligned} \tag{1.6}$$

Now, notice that

$$\left(\frac{a}{\sqrt{a^2+b^2}}\right)^2 + \left(\frac{b}{\sqrt{a^2+b^2}}\right)^2 = 1$$

This means that for some number (angle) β,

$$\cos\beta = \frac{a}{\sqrt{a^2+b^2}}, \quad \sin\beta = \frac{b}{\sqrt{a^2+b^2}}.$$

So, equation (1.6) may be rewritten

$$a\sin kx + b\cos kx = \sqrt{a^2+b^2}(\cos\beta\sin kx + \sin\beta\cos kx).$$

So that,

$$a\sin kx + b\cos kx = \sqrt{a^2+b^2}\ \ [\sin(\beta+kx)].$$

Or, equivalently,

$$a\sin kx + b\cos kx = \sqrt{a^2+b^2}\ \ \left(\sin k(x+\frac{\beta}{k})\right).$$

Evidently, the sum of the sine and cosine functions with matching frequencies and amplitudes a and b respectively is a sine function amplitude $\sqrt{a^2+b^2}$ shifted $\frac{\beta}{k}$ left.