Proof.

Let S_n be the statement $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{(n-k)} b^k$, where $a, b \in \mathbb{R}$ and $n = 1, 2, 3, \ldots$.

<u>Basis</u>. Verify that S_0 is true.

$$\text{LHS} = (a+b)^1 = a+b = \binom{1}{0} a^{1-0} b^0 + \binom{1}{1} a^{1-1} b^1 = \sum_{k=0}^{1} \binom{1}{k} a^{1-k} b^k = \text{RHS}.$$

<u>Inductive step</u>.

Suppose S_m is true for some $m \in \mathbb{N}$, i.e.

$$S_m : (a+b)^m = \sum_{k=0}^{m} \binom{m}{k} a^{(m-k)} b^k .$$

Show S_{m+1} is true, i.e.

$$S_{m+1} : (a+b)^{m+1} = \sum_{k=0}^{m+1} \binom{m+1}{k} a^{(m+1)-k} b^k .$$

1. LHS

2. $= (a+b)^{m+1}$

3. $= (a+b)\,(a+b)^m$

4. $= (a+b) \sum_{k=0}^{m} \binom{m}{k} a^{(m-k)} b^k$, using hypothesis

5. $= \sum_{k=0}^{m} \binom{m}{k} a^{(m+1)-k} b^k + \sum_{k=0}^{m} \binom{m}{k} a^{m-k} b^{k+1}$

6. $= \sum_{k=0}^{m} \binom{m}{k} a^{(m+1)-k} b^k + \sum_{k=1}^{m+1} \binom{m}{k-1} a^{m-(k-1)} b^k$

7.

$$= \binom{m}{0} a^{m+1-0} b^0 + \sum_{k=1}^{m} \binom{m}{k} a^{(m+1)-k} b^k + \sum_{k=1}^{m} \binom{m}{k-1} a^{(m+1)-k} b^k + \binom{m}{(m+1)-1} a^{(m+1)-(m+1)} b^{(m+1)}$$

8. $= \binom{m}{0} a^{m+1-0} b^0 + \sum_{k=1}^{m} \binom{m}{k} a^{(m+1)-k} b^k + \sum_{k=1}^{m} \binom{m}{k-1} a^{(m+1)-k} b^k + \binom{m}{m} a^0 b^{(m+1)}$

9. $= \binom{m}{0} a^{m+1} b^0 + \left(\sum_{k=1}^{m} \left[\binom{m}{k} + \binom{m}{k-1} \right] a^{(m+1)-k} b^k \right) + \binom{m}{m} a^0 b^{m+1}$, lemma

10. $= \binom{m+1}{0} a^{m+1} b^0 + \sum_{k=1}^{m} \binom{m+1}{k} a^{(m+1)-k} b^k + \binom{m+1}{m+1} a^0 b^{m+1}$, Pascal's formula

11. $= \sum_{k=0}^{m+1} \binom{m+1}{k} a^{(m+1)-k} b^k$

12. $=$ RHS

Conclusion. If S_m is true, then S_{m+1} is true. Since S_1 is true, S_n is true for all $n \in \mathbb{N}$.

$\therefore (a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{(n-k)} b^k$, where $a, b \in \mathbb{R}$ and $n = 1, 2, 3, \ldots$.

□