[06-02-21-T11]

Mathematical Induction

Mathematical induction is a method of proving that a statement is true for all positive integers. Here is how it works.

First, show that a statement we will call "statement-one" is true. Second, show that *if* statement-one is true, *then* statement-two must be true and *if* statement-two is true, *then* statement-three must be true, and *if* statement-three is true, *then* statement-four must be true, and so on without end.

Let's label statement-one as S_1 and statement-two as S_2 and so on. Then we can summarize the process described above.

First.

S_1 is true.

Second.

S_1 is true $\Longrightarrow$ S_2 is true,
S_2 is true $\Longrightarrow$ S_3 is true,
S_3 is true $\Longrightarrow$ S_4 is true,
$\ldots$

Third.

The statement S_n is true, where $n = 1, 2, 3, \ldots$.

The bad news is that it looks like the second part will take some serious time, since apparently one must prove infinitely many implications of the form *if this statement is true, then the next one is true.* That is, infinitely many implications of the form $S_k \Longrightarrow S_{k+1}$.

The good news is that we can accomplish this in a finite amount of time if we can prove that *if any statement-k is true than the next one, statement-(k+1) is must be true, too.* That is, if we can prove $S_k \Longrightarrow S_{k+1}$ for any $k = 1, 2, 3, \ldots$.

The following examples illustrate the proof technique of mathematical induction.

[EX1] Prove: $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$ for $n = 2, 3, 4, \ldots$.

Proof.

Let S_n be the statement $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$, $n = 1, 2, 3, \ldots$.

Basis. Verify that S_1 is true. LHS $= 1 = \frac{1\,(1+1)}{2} =$ RHS.

Inductive step. Show that *if* S_k is true *then* S_{k+1} must be true, $k \in \mathbb{Z}^+$.

Suppose S_k is true for some $k \in \mathbb{Z}^+$, i.e.

$S_k : 1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}$. [Note: statement S_k is known as the induction hypothesis.]

Show that then S_{k+1} must be true, i.e.

$S_{k+1} : 1 + 2 + 3 + \ldots + (k+1) = \frac{(k+1)\,((k+1)+1)}{2} = \frac{(k+1)\,(k+2)}{2}$.

LHS $= 1 + 2 + 3 + \ldots + k + (k+1)$

$= \frac{k(k+1)}{2} + (k+1)$ [Note: here we have used the hypothesis, statement S_k.]

$= \frac{k\,(k+1) + 2\,(k+1)}{2}$

$= \frac{(k+1)\,(k+2)}{2}$

$=$ RHS

Conclusion. If S_k is true, then S_{k+1} is true. Since S_1 is true, $S_1, S_2, S_3, \ldots$ are all true.

$\therefore$ S_n is true. $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$.

□

[EX2] Prove: $1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \ldots + n(n+1) = \frac{1}{3}\, n(n+1)(n+2)$ for $n = 1, 2, 3, \ldots$.
(J11-BA p. 88 Demonstration 1)

Proof.

Let S_n be the statement $1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \ldots + n(n+1) = \frac{1}{3}\, n(n+1)(n+2)$, $n = 1, 2, 3, \ldots$.

Basis. S_1 is true because LHS $= 1\cdot 2 = 2 = \frac{1\,(1+1)\,(1+2)}{3}$ = RHS.

Inductive step.

Suppose S_k is true for some $k \in \mathbb{Z}^+$, i.e.

$$S_k: 1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \ldots + k(k+1) = \frac{1}{3}\, k(k+1)(k+2).$$

Show S_{k+1} is true, i.e.

$$S_{k+1}: 1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \ldots + k(k+1) + (k+1)(k+2) = \frac{1}{3}\,(k+1)(k+2)(k+3).$$

$$\begin{aligned} \text{LHS} &= 1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \ldots + k\,(k+1) + (k+1)(k+2) \\ &= \frac{1}{3}\, k\,(k+1)(k+2) + (k+1)(k+2) \\ &= \frac{k^3}{3} + k^2 + \frac{2\,k}{3} + k^2 + 3\,k + 2 \\ &= \frac{1}{3}\,(k+1)(k+2)(k+3) \\ &= \text{RHS} \end{aligned}$$

Conclusion. If S_k is true, then S_{k+1} is true. Since S_1 is true, $S_1, S_2, S_3, \ldots$ are all true.

$\therefore\ 1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \ldots + n(n+1) = \frac{1}{3}\, n(n+1)(n+2)$.

□

[EX3] Bernoulli's Inequality (J11-BA p. 89 Demonstration 2)

Bernoulli's Inequality

Let $a > -1$ [Note: says same as $1 + a > 0$]

Thm: $(1 + a)^n \geq 1 + n\,a$, $n = 0, 1, 2, 3, \ldots$.

Proof.

Let S_n be the statement $(1 + a)^n \geq 1 + n\,a$, $n = 0, 1, 2, 3, \ldots$.

Basis. S_0 is true because $(1 + a)^0 = 1 \geq 1 + 0\,a$

Inductive step.

Suppose S_k is true for some $k = 0, 1, 2, \ldots$, i.e.

$S_k : (1 + a)^k \geq 1 + k\,a$

Show S_{k+1} is true, i.e.

$S_{k+1} : (1 + a)^{k+1} \geq 1 + (k + 1)\,a$

LHS $= (1 + a)^{k+1}$

$= (1 + a)^k (1 + a)$

$\geq (1 + k\,a)(1 + a)$ [using hypothesis]

$= 1 + k\,a + a + k\,a^2$

$\geq 1 + k\,a + a$ [$\because k\,a^2 \geq 0$]

$= 1 + (k + 1)\,a$

$=$ RHS

Conclusion. If S_k is true, then S_{k+1} is true. Since S_o is true, $S_0, S_1, S_2, \ldots$ are all true.

$\therefore (1 + a)^n \geq 1 + n\,a$, $n = 1, 2, 3, \ldots$, and $a > -1$.

□

[EX4] Prove: $a(b_1 + b_2 + b_3 + \ldots + b_n) = a\,b_1 + a\,b_2 + a\,b_3 + \ldots + a\,b_n$ for $n = 2, 3, 4, \ldots$.

Proof.

Let S_n be the statement $a(b_1 + b_2 + b_3 + \ldots + b_n) = a\,b_1 + a\,b_2 + a\,b_3 + \ldots + a\,b_n$ for $n = 2, 3, 4, \ldots$.

Basis. S_2 is true because $a(b_1 + b_2) = a\,b_1 + a\,b_2$

Inductive step.

Suppose S_k is true for some $k \in \mathbb{Z}^+$, i.e.

$$S_k : a(b_1 + b_2 + b_3 + \ldots + b_k) = a\,b_1 + a\,b_2 + a\,b_3 + \ldots + a\,b_k$$

Show S_{k+1} is true, i.e.

$$S_{k+1} : a(b_1 + b_2 + b_3 + \ldots + b_{k+1}) = a\,b_1 + a\,b_2 + a\,b_3 + \ldots + a\,b_{k+1}$$

LHS $= a\,(b_1 + b_2 + b_3 + \ldots + b_k + b_{k+1})$

$= a[(b_1 + b_2 + b_3 + \ldots + b_k) + b_{k+1}]$

$= a(b_1 + b_2 + b_3 + \ldots + b_k) + a\,b_{k+1}$

$= a\,b_1 + a\,b_2 + a\,b_3 + \ldots + a\,b_k + a\,b_{k+1}$ [using hypothesis]

$=$ RHS

Conclusion. If S_k is true, then S_{k+1} is true. Since S_2 is true, S_2, S_3, S_4, $\ldots$ are all true.

$\therefore a(b_1 + b_2 + b_3 + \ldots + b_n) = a\,b_1 + a\,b_2 + a\,b_3 + \ldots + a\,b_n$ for $n = 2, 3, 4, \ldots$.

□

[EX5] Prove: $\frac{1}{x-1} - \frac{1}{x+1} - \frac{2}{x^2+1} - \frac{4}{x^4+1} - \ldots - \frac{2^{n-1}}{x^{2^{n-1}}+1} = \frac{2^n}{x^{2^n}-1}$**, $n \in \mathbb{N}$, $n \geq 2$ (J10 p. 48 #4)**

[In book J10 at p. 48 #4, $\frac{1}{x-1} - \frac{1}{x+1} - \frac{2}{x^2+1} - \frac{4}{x^4+1} = \frac{8}{x^8-1}$]

Proof

Basis:

$$S_2 : \frac{1}{x-1} - \frac{1}{x+1} = \frac{2}{x^2-1}$$

$$\text{LHS} = \frac{1}{x-1} - \frac{1}{x+1} = \frac{2}{x^2-1} = \text{RHS}$$

Induction:

Suppose S_k is true

$$S_k : \frac{1}{x-1} - \frac{1}{x+1} - \frac{2}{x^2+1} - \ldots - \frac{2^{k-1}}{x^{2^{k-1}}+1} = \frac{2^k}{x^{2^k}-1}$$

Show S_{k+1} is true

$$S_{k+1} : \frac{1}{x-1} - \frac{1}{x+1} - \frac{2}{x^2+1} - \ldots - \frac{2^{k-1}}{x^{2^{k-1}}+1} - \frac{2^k}{x^{2^k}+1} = \frac{2^{k+1}}{x^{2^{k+1}}-1}$$

$$\text{LHS} = \frac{1}{x-1} - \frac{1}{x+1} - \frac{2}{x^2+1} - \ldots - \frac{2^{k-1}}{x^{2^{k-1}}+1} - \frac{2^k}{x^{2^k}+1}$$

$$= \frac{2^k}{x^{2^k}-1} - \frac{2^k}{x^{2^k}+1}$$

$$= \frac{2^{k+1}}{x^{2^{k+1}}-1}$$

=RHS

Since S_k is true implies S_{k+1} is true, and since S_2 is true, all of S_2, S_3, S_4, S_5, … are true.

$$\therefore \frac{1}{x-1} - \frac{1}{x+1} - \frac{2}{x^2+1} - \frac{4}{x^4+1} - \ldots - \frac{2^{n-1}}{x^{2^{n-1}}+1} = \frac{2^n}{x^{2^n}-1}$$

□

Proof.

Let S_n be the statement

Basis. S_1 is true because

Inductive step.

Suppose S_k is true for some $k \in \mathbb{Z}^+$, i.e.

S_k :

Show S_{k+1} is true, i.e.

S_{k+1} :

LHS =

= RHS

Conclusion. If S_k is true, then S_{k+1} is true. Since S_1 is true, S_1, S_2, S_3, … are all true.

∴

□