[11-12-09-T] Parts I, II, III

Trigonometric equations

Solving a trigonometric equation is complicated by two facts.

One, there are generally two numbers between zero and 2π for which the value of the function is identical. For example, you know that $\sin \frac{\pi}{6} = \frac{1}{2}$ and that $\sin(\pi - \frac{\pi}{6}) = \frac{1}{2}$. So, if asked to produce the solution of $\sin x = \frac{1}{2}$ for $0 \le x < 2\pi$, you would have to say that there are two values of x that make the equation true and that they are $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$.

Two, if x is allowed to range over all the real numbers, $x \in \mathbb{R}$, as opposed to x being restricted to the interval $[0, 2\pi)$, then there are infinitely many numbers that correspond to the point $\frac{\pi}{6}$ and to the point $\frac{5\pi}{6}$ and therefore have the same sine as do $\frac{\pi}{6}$ and $\frac{5\pi}{6}$. So, since
$\sin \frac{\pi}{6} = \frac{1}{2}$, $\sin(\frac{\pi}{6} + 2n\pi) = \frac{1}{2}$, too. And, since $\sin \frac{5\pi}{6} = \frac{1}{2}$, $\sin(\frac{5\pi}{6} + 2n\pi) = \frac{1}{2}$, too.

■ Part I Easy Trig Equations. These require no identities beyond those based on the radius, symmetry and periodicity of the unit circle (i.e. those on the first side of the identity sheet).

*In the following, a fundamental fact that you either "just know" or obtain using a machine, will be printed in **bold** face type. Every solution will begin with such a fact. Also, it is understood that $n \in \mathbb{Z}$.*

[1] Solve for $x \in \mathbb{R}$, if $\sin x = \frac{1}{2}$.

SOLUTION.

$\boldsymbol{\sin \frac{\pi}{6} = \frac{1}{2}}$ or $\sin(\pi - \frac{\pi}{6}) = \frac{1}{2}$. $\because \sin\theta = \sin(\pi - \theta)$ due to symmetry.

$\sin(\frac{\pi}{6} + 2n\pi) = \frac{1}{2}$ or $\sin(\frac{5\pi}{6} + 2n\pi) = \frac{1}{2}$. $\because \sin\theta = \sin(\theta + 2n\pi)$ due to periodicity.

$\therefore\ x = \frac{\pi}{6} + 2n\pi \bigvee x = \frac{5\pi}{6} + 2n\pi$.

[2] Solve for $x \in \mathbb{R}$, if $\cos x = \frac{\sqrt{3}}{2}$.

SOLUTION.

$\boldsymbol{\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}}$ or $\cos(\frac{-\pi}{6}) = \frac{\sqrt{3}}{2}$. $\because \cos\theta = \cos(-\theta)$ due to symmetry.

$\cos(\frac{\pi}{6} + 2n\pi) = \frac{\sqrt{3}}{2}$ or $\cos(\frac{-\pi}{6} + 2n\pi) = \frac{\sqrt{3}}{2}$. $\because \cos\theta = \cos(\theta + 2n\pi)$ due to periodicity.

$\therefore\ x = \frac{\pi}{6} + 2n\pi \bigvee x = \frac{-\pi}{6} + 2n\pi$.

[3] Solve for $x \in \mathbb{R}$, if $\sin x = \frac{-1}{2}$.

SOLUTION.

$\mathbf{\sin(\frac{\pi}{6}) = \frac{1}{2}}$ or so $-\sin(\frac{\pi}{6}) = -\frac{1}{2}$. $\because$ Algebra: $a = b \Longleftrightarrow -a = -b$.

$-\sin(\frac{\pi}{6}) = \sin(-\frac{\pi}{6})$. $\because -\sin\theta = \sin(-\theta)$. Symmetry.

$-\sin(\frac{\pi}{6}) = \sin(\pi + \frac{\pi}{6})$. $\because -\sin\theta = \sin(\pi + \theta)$. Symmetry.

$\sin(\frac{-\pi}{6} + 2n\pi) = -\frac{1}{2}$ or $\sin(\pi + \frac{\pi}{6} + 2n\pi) = -\frac{1}{2}$. $\because \sin\theta = \sin(\theta + 2n\pi)$ due to periodicity.

$\therefore\ x = \frac{-\pi}{6} + 2n\pi \bigvee x = \frac{7\pi}{6} + 2n\pi$.

We are happy when the equation may be solved exactly, because we do not need to bother to hunt up a calculator to acquire the fundamental fact. However, when only an approximate solution is possible, the only difference in the above procedure is that instead of consulting one's brain for the fundamental fact, one consults a machine.

[4] Solve for $x \in \mathbb{R}$, if $\cos x = 0.62$

SOLUTION.

$\mathbf{\cos 0.9021 \approx 0.6200}$, $\cos(-0.9021) = 0.6200$. $\because \cos\theta = \cos(-\theta)$

$\cos(0.9021 + 2n\pi) = 0.6200$ or $\cos(-0.9021 + 2n\pi) = -0.6200$. $\because \cos\theta = \cos(\theta + 2n\pi)$

$\therefore\ x = 0.9021 + 2n\pi \vee x = -0.9021 + 2n\pi$

[5] Solve for $x \in \mathbb{R}$, if $\sin x = -0.9781$.

SOLUTION.

$\mathbf{\sin(1.3611) \approx 0.9781}$, so $-\sin(1.3611) = -0.9781$. $\because$ Algebra.

$-\sin(1.3611) = \sin(-1.3611)$. $\because$ Symmetry.

$-\sin(1.3611) = \sin(\pi + 1.3611)$. $\because$ Symmetry.

$\sin(-1.3611 + 2n\pi) = -0.9781$ or $\sin(\pi + 1.3611 + 2n\pi) = -0.9781$. $\because$ Periodicity.

$\therefore\ x = -1.3611 + 2n\pi \vee x = \pi + 1.3611 + 2n\pi$.

Usually, a little algebra will be necessary to rewrite the equation in the form of those above: trigfuntion(x) = number.

[6] Solve for $x \in \mathbb{R}$, if $2 \sin x + \sqrt{3} = 0$.

SOLUTION.

$2 \sin x + \sqrt{3} = 0 \Longleftrightarrow \sin x = \frac{-\sqrt{3}}{2}$. Algebra.

$\mathbf{\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}}$, so $-\sin(\frac{\pi}{3}) = -\frac{\sqrt{3}}{2}$. $\because$ Algebra.

$-\sin(\frac{\pi}{3}) = \sin(-\frac{\pi}{3})$. $\because$ Symmetry.

$-\sin(\frac{\pi}{3}) = \sin(\pi + \frac{\pi}{3})$. $\because$ Symmetry.

$\sin(-\frac{\pi}{3} + 2 n \pi) = \frac{-\sqrt{3}}{2}$ or $\sin(\frac{4\pi}{3} + 2 n \pi) = \frac{-\sqrt{3}}{2}$. $\because$ Periodicity.

$\therefore\ x = -\frac{\pi}{3} + 2 n \pi + 2 n \pi \bigvee x = \frac{4\pi}{3} + 2 n \pi$.

[7] Solve for $x \in \mathbb{R}$, if $\sin^2 x + 7 \sin x + 6 = 0$.

SOLUTION.

$\sin^2 x + 7 \sin x + 6 = 0 \Longleftrightarrow (\sin x + 1)(\sin x + 6) = 0$. Algebra.

This implies,

$\sin x = -1$, $\sin x = -6$.

$\mathbf{\sin(\frac{\pi}{2}) = 1}$, so $-\sin(\frac{\pi}{2}) = -1$. $\because$ Algebra.

$-\sin(\frac{\pi}{2}) = \sin(\frac{-\pi}{2}) = \sin(\frac{3\pi}{2})$. $\because$ Symmetry.

$-\sin(\frac{\pi}{2}) = \sin(\pi + \frac{\pi}{2}) = \sin(\frac{3\pi}{2})$. $\because$ Symmetry.

These both come to the same thing: $\sin(\frac{3\pi}{2})$

$\sin(\frac{3\pi}{2} + 2 n \pi) = -1$. $\because$ Periodicity.

Then for the other root of $\sin^2 x + 7 \sin x + 6 = 0$.

There is no number whose sine is -6. $\because$ $-1 \le \sin\theta \le 1$. Unit circle is radius 1. So, fact

$\mathbf{\{x : \sin x = -6\} = \emptyset}$.

$\therefore\ x = \frac{3\pi}{2} + 2 n \pi$.

Remember that the period of the tangent and cotangent functions is π.

[8] Solve for $x \in \mathbb{R}$, if $\tan x = -\sqrt{3}$.

SOLUTION.

$\mathbf{tan}\left(\frac{\pi}{3}\right) = \sqrt{3}$, so $-\tan\left(\frac{\pi}{3}\right) = -\sqrt{3}$.

$-\tan\left(\frac{\pi}{3}\right) = \tan\left(-\frac{\pi}{3}\right)$. $\because -\tan\theta = \tan(-\theta)$. Symmetry.

$-\tan\left(\frac{\pi}{3}\right) = \tan\left(\pi - \frac{\pi}{3}\right)$. $\because -\tan\theta = \tan(\pi - \theta)$. Symmetry.

$\tan\left(\frac{-\pi}{3} + n\pi\right) = -\sqrt{3}$ or $\sin\left(\pi - \frac{\pi}{3} + n\pi\right) = -\sqrt{3}$. $\because \tan\theta = \tan(\theta + n\pi)$ due to periodicity.

$\therefore\ x = \frac{-\pi}{3} + n\pi \vee x = \frac{2\pi}{3} + n\pi$. Notice the period is not 2π.

■ Equations when the frequency is not equal to 1.

In all of the examples worked so far, the frequency of the function has been equal to 1. Formally, a frequency other than 1 is handled in the same manner as the examples above.

[9] Solve for $x \in \mathbb{R}$, if $\sin 4x = \frac{-1}{2}$.

SOLUTION.

$\mathbf{sin}\left(\frac{\pi}{6}\right) = \frac{1}{2}$, so $-\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$.

$-\sin\left(\frac{\pi}{6}\right) = \sin\left(-\frac{\pi}{6}\right)$.

$-\sin\left(\frac{\pi}{6}\right) = \sin\left(\pi + \frac{\pi}{6}\right)$.

$\sin\left(\frac{-\pi}{6} + 2n\pi\right) = -\frac{1}{2}$ or $\sin\left(\pi + \frac{\pi}{6} + 2n\pi\right) = -\frac{1}{2}$.

So,

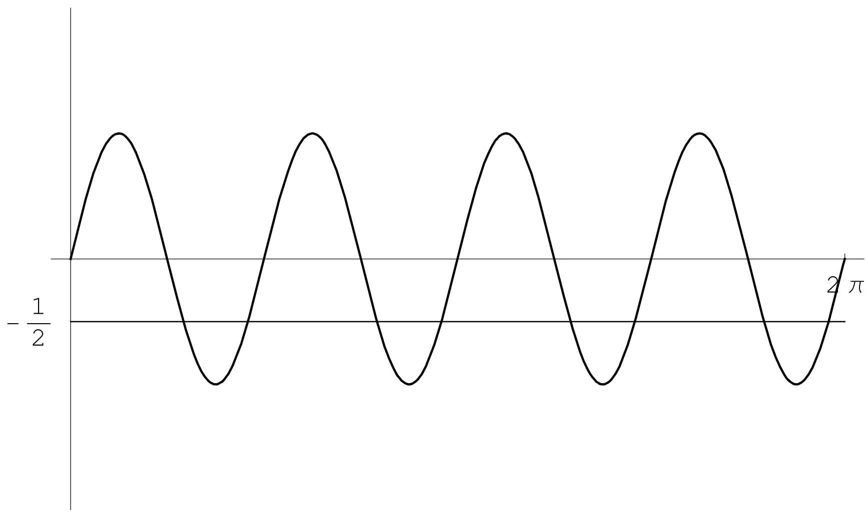
$x = \frac{-\pi}{6} + 2n\pi \vee 4x = \frac{7\pi}{6} + 2n\pi$.

$\therefore\ x = \frac{-\pi}{24} + \frac{n\pi}{2} \vee x = \frac{7\pi}{24} + \frac{n\pi}{2}$.

Notice that the period of this version of the sine function is $\frac{\pi}{2}$. This means that the function $f(x) = \sin 4x$ completes 1 full cycle over an interval of length $\frac{\pi}{2}$. And, this implies that over the natural period, 2π, of the sine function, the function f completes 4 full cycles. How many solutions of $\sin 4x = \frac{-1}{2}$ would one expect in $[0, 2\pi)$?

If asked to find all solutions of $f(x) = \sin 4x$ for $0 \le x < 2\pi$, one would expect number of values of x would be 8. To find them all, let n take the values $n = 0, 1, 2, \ldots$ until $x \ge 2$ pi. In example [9],

n	$\frac{-\pi}{24} + \frac{n\pi}{2}$	$\frac{7\pi}{24} + \frac{n\pi}{2}$
0	$\frac{-\pi}{24} < 0$	$\mathbf{\frac{7\pi}{24}}$
1	$\mathbf{\frac{11\pi}{24}}$	$\mathbf{\frac{19\pi}{24}}$
2	$\mathbf{\frac{23\pi}{24}}$	$\mathbf{\frac{31\pi}{24}}$
3	$\mathbf{\frac{35\pi}{24}}$	$\mathbf{\frac{43\pi}{24}}$
4	$\mathbf{\frac{47\pi}{24}}$	$\frac{55\pi}{24} > 2\pi$



- **. . . and if the phase shift is not equal to 0.**

[10] Solve for $x \in \mathbb{R}$, if $\sin(x - \frac{\pi}{4}) = \frac{-1}{2}$.

SOLUTION.

$\mathbf{\sin(\frac{\pi}{6}) = \frac{1}{2}}$, so $-\sin(\frac{\pi}{6}) = -\frac{1}{2}$.

$-\sin(\frac{\pi}{6}) = \sin(-\frac{\pi}{6})$.

$-\sin(\frac{\pi}{6}) = \sin(\pi + \frac{\pi}{6})$.

$\sin(\frac{-\pi}{6} + 2n\pi) = -\frac{1}{2}$ or $\sin(\pi + \frac{\pi}{6} + 2n\pi) = -\frac{1}{2}$.

So,

$x - \frac{\pi}{4} = \frac{-\pi}{6} + 2n\pi \vee x - \frac{\pi}{4} = \frac{7\pi}{6} + 2n\pi$.

$x = \frac{-\pi}{6} + \frac{\pi}{4} + 2n\pi \vee x = \frac{7\pi}{6} + \frac{\pi}{4} + 2n\pi$.

$\therefore\ x = \frac{\pi}{12} + 2n\pi \vee x = \frac{17\pi}{12} + 2n\pi$.

■ **. . . and if frequency not equal to 1 and phase shift not equal to 0.**

[11] Solve for $x \in \mathbb{R}$, if $\sin \frac{1}{2}(x - \frac{\pi}{3}) = \frac{-\sqrt{2}}{2}$.

SOLUTION.

$\mathbf{\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}}$, so $-\sin(\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}$.

$-\sin(\frac{\pi}{4}) = \sin(-\frac{\pi}{4})$.

$-\sin(\frac{\pi}{4}) = \sin(\pi + \frac{\pi}{4})$.

$\sin(\frac{-\pi}{4} + 2n\pi) = -\frac{\sqrt{2}}{2}$ or $\sin(\pi + \frac{\pi}{4} + 2n\pi) = -\frac{\sqrt{2}}{2}$.

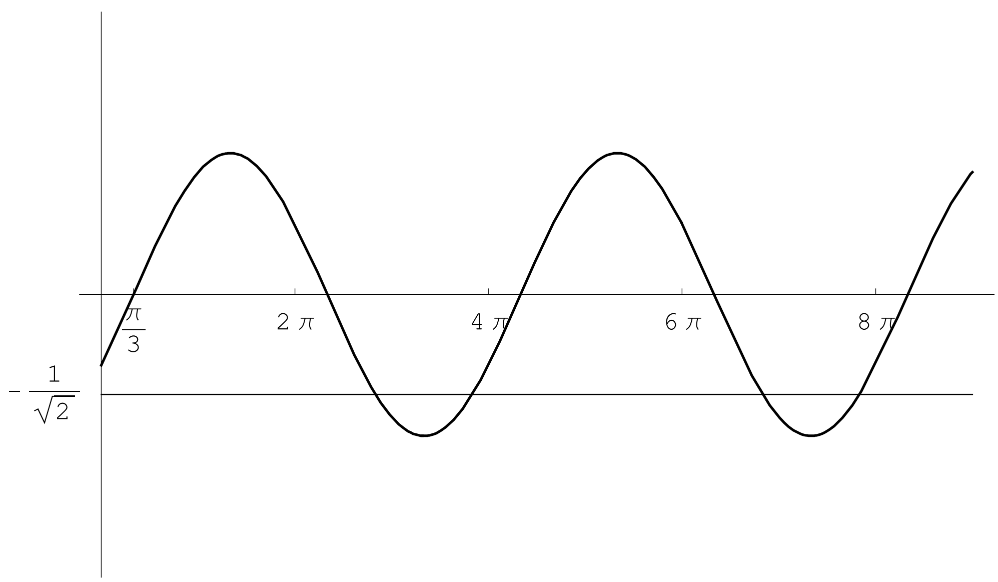
So,

$\frac{1}{2}(x - \frac{\pi}{3}) = \frac{-\pi}{4} + 2n\pi \vee \frac{1}{2}(x - \frac{\pi}{3}) = \frac{5\pi}{4} + 2n\pi$.

$x = \frac{-\pi}{2} + \frac{\pi}{3} + 4n\pi \vee x = \frac{5\pi}{2} + \frac{\pi}{3} + 4n\pi$.

$\therefore\ x = -\frac{\pi}{6} + 4n\pi \vee x = \frac{17\pi}{6} + 4n\pi$.

The graph of this follows.



■ Part II Harder Trig Equations. These require identities from addition theorems.

It is usually an advantage to rewrite a trigonometric equation so that only one trigonometric function is involved.

[11] Solve for $x \in \mathbb{R}$, if $\sec^2 x + \tan^2 x = 3$.

SOLUTION.

$\sec^2 x + \tan^2 x = 3$
$\Longleftrightarrow \sec^2 x + (\sec^2 x - 1)$
$\Longleftrightarrow 2\sec^2 x - 1 = 3$
$\Longleftrightarrow \sec^2 x = 2$
$\Longleftrightarrow \sec x = \sqrt{2} \bigvee \sec x = -\sqrt{2}$

Case 1. $\sec x = \sqrt{2}$.

$\mathbf{sec}\ (\frac{\pi}{4}) = \sqrt{2}$.

$\sec(\frac{\pi}{4}) = \sec(-\frac{\pi}{4})$.

$\sec(\frac{\pi}{4} + 2n\pi) = \sqrt{2}$, $\sec(\frac{-\pi}{4} + 2n\pi) = \sqrt{2}$.

$x = \frac{\pi}{4} + 2n\pi,\ x = \frac{-\pi}{4} + 2n\pi$.

Case 2. $\sec x = -\sqrt{2}$.

$\mathbf{sec}\ (\frac{\pi}{4}) = \sqrt{2}$.

$-\sec\ (\frac{\pi}{4}) = -\sqrt{2}$.

$-\sec(\frac{\pi}{4}) = \sec(\pi - \frac{\pi}{4} + 2n\pi) = \sec(\frac{3\pi}{4} + 2n\pi)$,
$-\sec(\frac{\pi}{4}) = \sec(\pi + \frac{\pi}{4} + 2n\pi) = \sec(\frac{5\pi}{4} + 2n\pi)$.

$\sec(\frac{3\pi}{4} + 2n\pi) = -\sqrt{2}$, $\sec(\frac{5\pi}{4} + 2n\pi) = -\sqrt{2}$.

$x = \frac{3\pi}{4} + 2n\pi \bigvee x = \frac{5\pi}{4} + 2n\pi$.

$\therefore\ x = \frac{\pi}{4} + \frac{n\pi}{2}$.

[12] Solve for $x \in \mathbb{R}$, if $\sin x - \cos x = 1$.

SOLUTION.

We can rewrite the equation in terms of a single function by using $\sin^2 \theta + \cos^2 \theta = 1$.

$\sin x - \cos x = 1$
$\Longleftrightarrow \sin x = \cos x + 1$
$\Longleftrightarrow \sin^2 x = \cos^2 x + 2\cos x + 1$
$\Longleftrightarrow 1 - \cos^2 x = \cos^2 x + 2\cos x + 1$
$\Longleftrightarrow -\cos^2 x = \cos^2 x + 2\cos x$
$\Longleftrightarrow 2\cos^2 x + 2\cos x = 0$
$\Longleftrightarrow 2\cos x(\cos x + 1) = 0$
$\Longleftrightarrow \cos x = 0 \;\text{ or }\; \cos x = -1$

Case 1. $\cos x = 0$.

$$x = \frac{\pi}{2} + 2n\pi,\ x = \frac{3\pi}{4} + 2n\pi.$$

Case 2. $\cos x = -1$.

$$x = \pi + 2n\pi.$$

Since we squared both sides, we may have introduced extraneous roots. So, we must check each possible solution in the original equation.

If $x = \frac{\pi}{2} + 2n\pi$, $\sin x = 1$, $\cos x = 0$, $\sin x - \cos x = 1$. (OK)

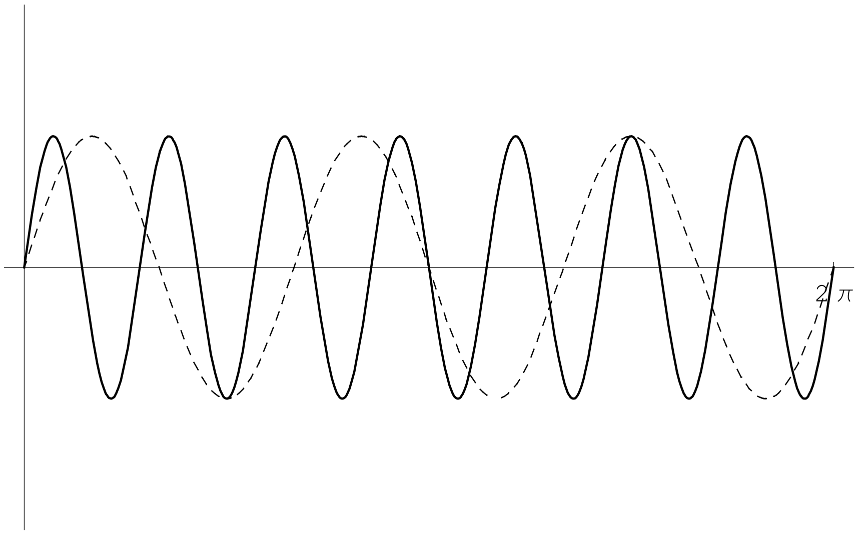
If $x = \frac{3\pi}{2} + 2n\pi$, $\sin x = -1$, $\cos x = 0$, $\sin x - \cos x = -1$. (NO)

If $x = \pi + 2n\pi$, $\sin x = 0$, $\cos x = -1$, $\sin x - \cos x = 1$. (OK)

$\therefore\ x = \frac{\pi}{2} + 2n\pi \bigvee \pi + 2n\pi$.

■ Part III Trig Equations involving functions having different periods.

Considered graphically, the equations so far treated have involved finding the intersections of a horizontal line and a trigonometric curve. The equation $\sin 3x = \sin 7x$ is solved when we know the points of intersection of the two curves. Below, $\sin 3x$ is the dashed line, $\sin 7x$ the solid line.



[13] Solve for $x \in \mathbb{R}$, if $\sin 3x = \sin 7x$.

SOLUTION.

$\sin 3x = \sin(\pi - 3x)$. $\because \sin\theta = \sin(\pi - \theta)$ due to symmetry.

So, $\sin 3x = \sin 7x$ if either

$$7x = 3x + 2n\pi \quad \text{or} \quad 7x = (\pi - 3x) + 2n\pi$$

$\therefore\ x = \frac{n\pi}{2} \vee x = \frac{\pi}{10} + \frac{n\pi}{5}$.

[14] Solve for $x \in \mathbb{R}$, if $\cos 5x = -\cos 2x$.

SOLUTION.

$-\cos 2x = \cos(\pi - 2x)$, $-\cos 2x = \cos(\pi + 2x)$.

Also,

If $\cos 5x = -\cos 2x$, then

$$5x = \pi - 2x + 2n\pi \quad \text{or} \quad 5x = \pi + 2x + 2n\pi$$

So that,

$$7x = \pi + 2n\pi \quad \text{or} \quad 3x = \pi + 2n\pi$$

$$x = \frac{\pi}{7} + \frac{2n\pi}{7} \quad \text{or} \quad x = \frac{\pi}{3} + \frac{2n\pi}{3}.$$

So, $\sin 3x = \sin 7x$ if either

$$7x = 3x + 2n\pi \quad \text{or} \quad 7x = (\pi - 3x) + 2n\pi$$

Since $\cos 5x = \cos(-5x)$, we also consider

$$-5x = \pi - 2x + 2n\pi \quad \text{or} \quad -5x = \pi + 2x + 2n\pi$$

$$-3x = \pi - 2n\pi \quad \text{or} \quad -7x = \pi + 2n\pi$$

$$x = -\frac{\pi}{3} - 2n\pi \quad \text{or} \quad 7x = -\frac{\pi}{7} - 2n\pi$$

$$\therefore\ x = \frac{\pi}{7} + \frac{2n\pi}{7} \vee -\frac{\pi}{7} + 2n\pi \vee x = \frac{\pi}{3} + \frac{2n\pi}{3} \vee x = -\frac{\pi}{3} + 2n\pi.$$

[15] Solve for $x \in \mathbb{R}$, if $\sin 2x = \cos 3x$.

SOLUTION.

$\cos 3x = \sin(\frac{\pi}{2} - 3x)$, $\cos(3x) = \cos(-3x) = \sin(\frac{\pi}{2} + 3x)$.

Then,

$$2x = \frac{\pi}{2} - 3x + 2n\pi \quad \text{or} \quad 2x = \frac{\pi}{2} + 3x + 2n\pi$$

So that,

$$5x = \frac{\pi}{2} + 2n\pi \quad \text{or} \quad -x = \frac{\pi}{2} + 2n\pi$$

$\therefore x = \frac{\pi}{10} + \frac{2n\pi}{5} \quad \text{or} \quad x = \frac{-\pi}{2} + 2n\pi.$