[11-10-30-T11]

Approximating the logarithm

$\ln \frac{x+1}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} \cdots\right)$ the equality is exact for $|x| < 1$. And it is a pretty series.

Let's compute an approximation of $\ln 2$ using first two terms of the series and then three terms.

To find a suitable x, let $\frac{x+1}{1-x} = 2$. Then $x = \frac{1}{3}$.

Using two terms,

$$\ln 2 = \ln \frac{\frac{1}{3}+1}{1-\frac{1}{3}} \approx 2\left(\frac{1}{3} + \frac{\left(\frac{1}{3}\right)^3}{3}\right) = 2\left(\frac{1}{3} + \frac{1}{81}\right) = 2\left(\frac{27+1}{81}\right) = \frac{56}{81} = 0.691358.$$

Compare this to the calculator value $\ln 2 \approx 0.693147$, $0.693147 - 0.691358 = 0.00178898$. Not bad for just two terms. The first difference is in the thousandths place. We are within 0.0018, with hardly any computational effort

Using three terms,

$$\ln 2 = \ln \frac{\frac{1}{3}+1}{1-\frac{1}{3}} \approx 2\left(\frac{1}{3} + \frac{\left(\frac{1}{3}\right)^3}{3} + \frac{\left(\frac{1}{3}\right)^5}{5}\right) = 2\left(\frac{1}{3} + \frac{1}{81} + \frac{1}{1215}\right) = 0.693004.$$

Compare this to the calculator value $\ln 2 \approx 0.693147$, $0.693147 - 0.693004 = 0.000142885$. The first difference is in the ten-thousandths place. We are within 0.00015.

In Grade 12 calculus, you will learn how to find series representations such as this one for the logarithm function, the trigonometric functions, and some other functions.