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[11-10-28-T11 REV2]

Trigonometry

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■ Radian measure

We are accustomed to measuring angles in degrees, but the natural measure is the radian.

DEFINITION. Let  $\overrightarrow{OP}$  name the ray of length  $r$  which is free to rotate about fixed point  $O$ . When point  $P$  has traced a circular arc of length equal to  $r$ , then we say  $\overrightarrow{OP}$  has rotated through an angle of one radian.

We determine the radian measure of any angle  $\theta$  of rotation by dividing the corresponding arc length  $s$  by the length  $r$  of the rotated ray. That is,  $\theta = \frac{s}{r}$ . The result is the size of  $\theta$  in radian measure. Note that the radian is a dimensionless unit, since the units of length in  $\frac{s}{r}$  cancel.

The equation  $\theta = \frac{s}{r}$  provides a formula for the length of an arc,  $s = r\theta$ . Notice that when  $s$  equals the circumference of a circle radius  $r$ ,  $s = r\theta$  becomes  $2\pi r = r\theta$  which tells us that one complete revolution (the central angle of the full circle), is  $2\pi$ . Since in degrees the central angle of the full circle is  $360^\circ$ , we have the conversion  $360^\circ = 2\pi$  radians or  $180^\circ = \pi$  rad. This means  $1 \text{ rad} = \frac{180^\circ}{\pi} \approx 57^\circ$ .

The area  $A_s$  of a sector of a circle radius  $r$ , arc length  $s$ , and central angle  $\theta$  can be found by noting that  $\frac{A_s}{\pi r^2} = \frac{s}{2\pi r}$ . Then  $A_s = \frac{1}{2} r s \iff A_s = \frac{1}{2} r^2 \theta$ . When  $\theta = 2\pi$ ,  $A_s = \frac{1}{2} r^2 (2\pi) = \pi r^2$ , the area of the circle radius  $r$ .

One caution, the formulae  $s = r\theta$  and  $A_s = \frac{1}{2} r^2 \theta$  are valid only in radian measure.

■ The Unit Circle

The unit circle is centered at origin and has radius equal to 1. The circumference of the unit circle is  $2\pi$ . For any circle,  $s = r\theta$ , where  $s =$  arc length,  $r =$  radius,  $\theta =$  central angle radians. On the unit circle,  $s = \theta$ . Positive angles (arc lengths) are measured counter-clockwise, and negative angles (arc lengths) clockwise, from the positive horizontal axis.

Any real number  $x$  corresponds to an arc length of  $s = x$ . So, the real number  $x$  is associated with a particular point on the unit circle. To avoid confusion, we usually label the horizontal axis  $u$  and the vertical axis  $v$ , instead of  $x$  and  $y$ . In general, real number  $x$  corresponds to point  $P(u, v)$ .

EXAMPLE 1. Find the  $u$ - $v$  coordinates of the point on the unit circle corresponding to the real number  $\frac{\pi}{2}$ . Solution. The number  $\frac{\pi}{2}$  is one-quarter of the unit circle's circumference of  $2\pi$ , so  $\frac{\pi}{2}$  corresponds to the point  $(0, 1)$  at the top of the circle.

EXAMPLE 2. Locate the point on the unit circle corresponding to the real number  $-\frac{\pi}{3}$ . Solution. The number  $\frac{\pi}{3}$  is  $\frac{1}{6}$  of  $2\pi$ , so place the point  $\frac{1}{6}$  of the way around the circle in the clockwise (negative) direction.

Since the circumference of the unit circle is  $2\pi$ , the real numbers  $x, x + 2\pi, x + 4\pi, \dots, x + 2n\pi$ ,  $n \in \mathbb{Z}$  all end up at precisely the same point on the unit circle. The coincidence of the points corresponding to  $x, x + 2\pi, x + 4\pi, \dots, x + 2n\pi$  is due to the *periodicity* of the unit circle. It is also because of this periodicity that the function from the set  $\mathbb{R}$  to the set of points on the unit circle fails to be a 1-1 function.

NOTE. From this time on, we will say "circle" to mean the unit circle and "length of the circle" to mean the circumference  $2\pi$  of the unit circle.

### The unit circle in the $uv$ -plane.

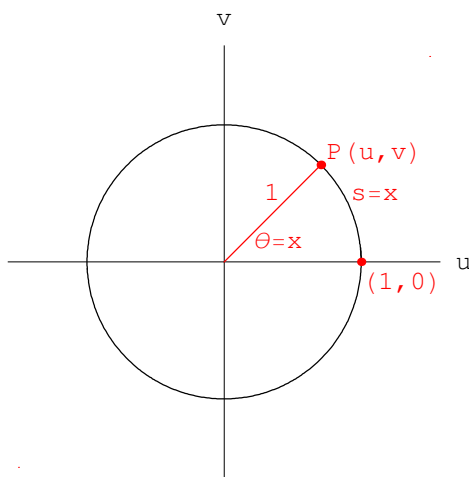


Figure 1

The real number  $x$  corresponds to the point  $P$  whose coordinates are  $u$  and  $v$ . Since  $s = \theta$ ,  $x$  is also the size of the central angle  $\theta$ . Figure 1 illustrates these relationships.

## ■ Symmetries of the unit circle

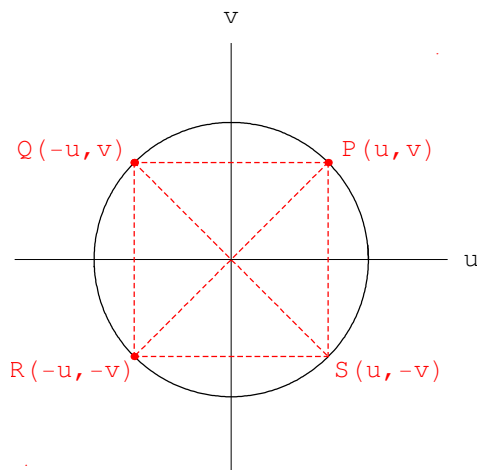


Figure 2

The symmetries shown in Figure 2 are the basis of many useful theorems in trigonometry. Notice that points P and Q are symmetric with respect to the v-axis, points P and S are symmetric with respect to the u-axis. Also note that points P and R are symmetric with respect to the origin. There are several other symmetries involving points P, Q, R, S that the reader will note.

Convince yourself that if P is the point on the circle corresponding to the real number  $x$ , then R corresponds to  $x \pm \pi$ , S corresponds to  $-x$ , and Q corresponds to  $\pi - x$ . For example, if P corresponds to  $x = \frac{\pi}{3}$ , the Q corresponds to  $x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ . Similarly, S would correspond to  $\frac{\pi}{3} + \pi = \frac{4\pi}{3}$ .

EXAMPLE 3. The point T on the circle corresponding to  $\frac{3}{4}\pi$  is in quadrant (II), because  $\frac{\pi}{2} < \frac{3\pi}{4} < \pi$ . The point T' that is symmetric to T with respect to the origin is a distance  $\pi + \frac{3\pi}{4} = \frac{7\pi}{4}$  around the circle. Since  $\frac{3\pi}{2} < \frac{7\pi}{4} < 2\pi$ , T' is in quadrant IV.

NOTE. Since it becomes tiresome to repeat the phrase "the point on the circle corresponding to the real number  $x$ ", we will from now on say simply "the point  $x$ ". So, instead of "the point on the circle corresponding to the real number  $\frac{\pi}{6}$ ", we say "the point  $\frac{\pi}{6}$ ".

EXAMPLE 4. Name the point symmetric to the point  $\frac{\pi}{6}$  with respect to (a) the u-axis, (b) the v-axis, and (c) the origin. Solution. (a)  $-\frac{\pi}{6}$ , (b)  $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$ , and (c)  $\pi + \frac{\pi}{6} = \frac{7\pi}{6}$ . Note that if we were to insist on measuring arc length in the positive direction, then we would answer (a) by saying  $2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$ .

■ Assign MT [1.1] here.

### *Special relationships (using a little geometry)*

■ The coordinates of the points corresponding to the numbers  $\frac{\pi}{6}$ ,  $\frac{\pi}{4}$ ,  $\frac{\pi}{3}$ .

The points on the unit circle at arc lengths  $\frac{\pi}{6}$ ,  $\frac{\pi}{3}$  correspond to central angles of the same size,  $\frac{\pi}{6}$ ,  $\frac{\pi}{3}$ . The central angles are equivalent to angles of  $\frac{\pi}{6} = 30^\circ$ ,  $\frac{\pi}{3} = 60^\circ$  in the more familiar sexagesimal (degree-minute-second) system. Thus, we wish to know the lengths  $u$  and  $v$  in figure 3.

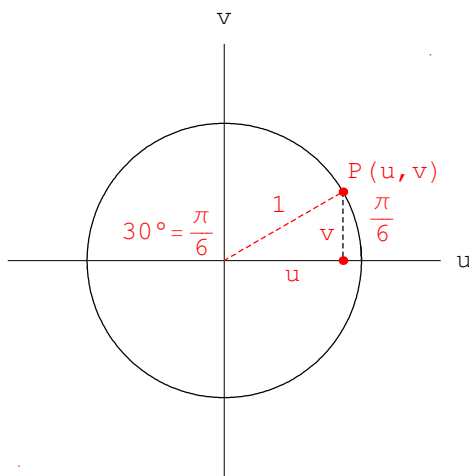


Figure 3

### ■ 30-60-90 Triangle

Draw an equilateral triangle of side 2 as in Figure 4. Construct angle bisector PQ. PQ results in a 30 – 60 – 90 triangle, hypotenuse length 2 and legs length 1 and  $x$ .

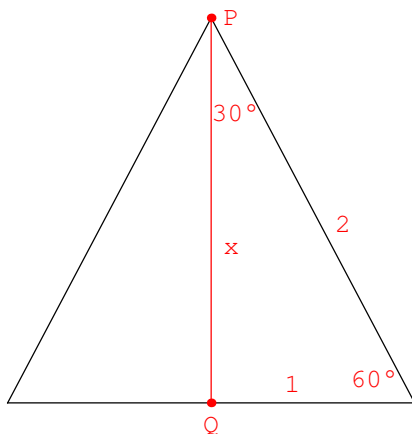


Figure 4

By the theorem of Pythagoras,  $x = \sqrt{3}$ . Dividing all sides by 2, produces a similar 30 – 60 – 90 triangle of sides  $\frac{1}{2}$ ,  $\frac{\sqrt{3}}{2}$ , 1. The later triangle is congruent to the one in Figure 3.

We now know that in Figure 3,  $u = \frac{\sqrt{3}}{2}$ ,  $v = \frac{1}{2}$ . That is,

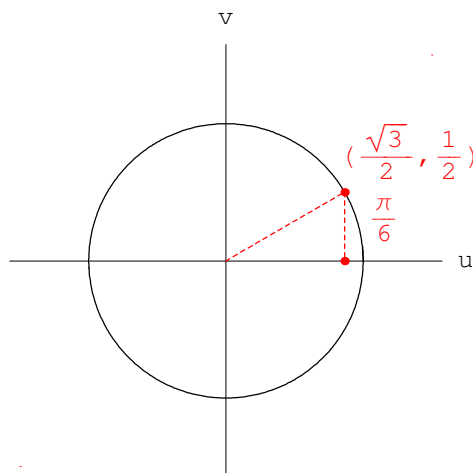


Figure 5

#### ■ 45-45-90 Triangle

Drawing a square side 2, then reasoning as above, will convince the reader that the coordinates of the point corresponding to  $\frac{\pi}{4}$  are  $u = \frac{\sqrt{2}}{2}$ ,  $v = \frac{\sqrt{2}}{2}$ .

It would be wise to know by heart the coordinates of the points at  $\frac{\pi}{6}$ ,  $\frac{\pi}{4}$ ,  $\frac{\pi}{3}$ . The reader cannot help but know the equally important coordinates at the intersections of the circle with the axes.

#### ■ $u^2 + v^2 = 1$

Applying the theorem of Pythagoras to the unit circle.

#### ■ Definitions. The trigonometric functions

DEFINITION. Let the point  $P(u, v)$  on the unit circle be the point that corresponds to the real number  $x$ . Then, the cosine function is defined by  $\cos x = u$ , the sine function is defined by  $\sin x = v$ . The tangent function is defined by  $\tan x = \frac{\sin x}{\cos x}$ ,  $\cos x \neq 0$ . We define the secant, cosecant, cotangent functions as respectively  $\sec x = \frac{1}{\cos x}$ ,  $\cos x \neq 0$ ,  $\csc x = \frac{1}{\sin x}$ ,  $\sin x \neq 0$ ,  $\cot x = \frac{\cos x}{\sin x}$ ,  $\sin x \neq 0$ . The sine, cosine, tangent, secant, and cosecant are collectively called the trigonometric functions. □

By virtue of the definitions above, many of the qualities of the trigonometric functions are due to the periodicity and symmetry of the circle. The following facts will be useful. They should be understood and remembered. For  $\theta \in \mathbb{R}$ , the following are true for all  $\theta$  for which they are defined.

$$[4] \quad -1 \leq \sin \theta \leq 1, \quad -1 \leq \cos \theta \leq 1$$

$$[5] \quad \sin^2 \theta + \cos^2 \theta = 1$$

$$[6] \quad \tan^2 \theta + 1 = \sec^2 \theta$$

$$[7] \quad 1 + \cot^2 \theta = \csc^2 \theta$$

Using the periodicity the unit circle

$$[8] \quad \begin{aligned} \sin(\theta + 2n\pi) &= \sin \theta \\ \cos(\theta + 2n\pi) &= \cos \theta \\ \tan(\theta + n\pi) &= \tan \theta \end{aligned}$$

Using the symmetry of the unit circle

$$[9] \quad \begin{aligned} \sin(-\theta) &= -\sin \theta \\ \cos(-\theta) &= \cos(\theta) \\ \tan(-\theta) &= -\tan \theta \end{aligned}$$

$$[11] \quad \begin{array}{ll} \sin(\pi - \theta) = \sin \theta & \sin(\pi + \theta) = -\sin \theta \\ \cos(\pi - \theta) = -\cos \theta & \cos(\pi + \theta) = -\cos \theta \\ \tan(\pi - \theta) = -\tan \theta & \tan(\pi + \theta) = \tan \theta \end{array}$$

There are several more facts that are nearly immediate. To understand them, we ought first to consider two points symmetric with respect to the line  $u = v$ . A little geometry will convince the reader that the coordinates of these two points are as shown in Figure 6.

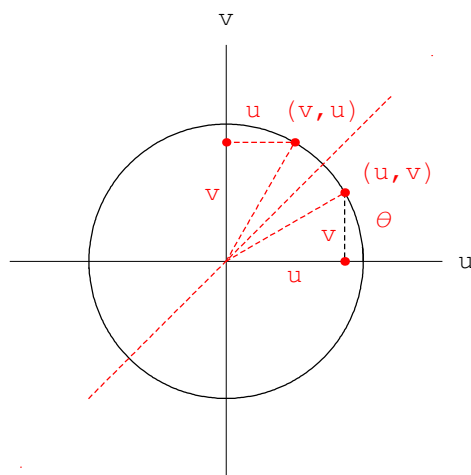


Figure 6

The identities below in the left column are a consequence of this geometry. The identities in the right column follow from those in the left by noting the points  $\frac{\pi}{2} - \theta$  and  $\frac{\pi}{2} + \theta$  are symmetric with respect to the  $v$ -axis.

$$\begin{array}{ll}
 [10] \quad \sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta & \sin\left(\frac{\pi}{2} + \theta\right) = \cos \theta \\
 \cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta & \cos\left(\frac{\pi}{2} + \theta\right) = -\sin \theta \\
 \tan\left(\frac{\pi}{2} - \theta\right) = \frac{1}{\tan \theta} & \tan\left(\frac{\pi}{2} + \theta\right) = -\frac{1}{\tan \theta}
 \end{array}$$

### ■ Examples A

$$[1.1] \quad \cos \frac{-\pi}{3} = \cos \frac{\pi}{3} = \frac{1}{2}; \quad \sin \frac{-\pi}{3} = -\sin \frac{\pi}{3} = \frac{-\sqrt{3}}{2}; \quad \tan \frac{-\pi}{3} = \frac{\frac{-\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}$$

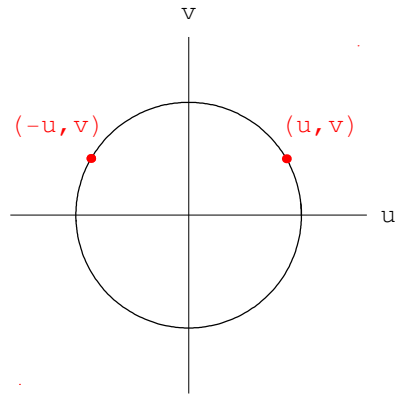
$$\begin{aligned}
 [1.2] \quad \cos \frac{5\pi}{6} &= \cos\left(\pi - \frac{\pi}{6}\right) = -\cos \frac{\pi}{6} = \frac{-\sqrt{3}}{2}; \quad \sin \frac{5\pi}{6} = \sin\left(\pi - \frac{\pi}{6}\right) = \sin \frac{\pi}{6} = \frac{1}{2}; \\
 \tan \frac{5\pi}{6} &= \frac{\frac{1}{2}}{\frac{-\sqrt{3}}{2}} = \frac{1}{-\sqrt{3}} = \frac{\sqrt{3}}{3}.
 \end{aligned}$$

$$\begin{aligned}
 [1.3] \quad \cos \frac{5\pi}{4} &= \cos\left(\pi + \frac{\pi}{4}\right) = -\cos \frac{\pi}{4} = \frac{-\sqrt{2}}{2}; \quad \sin \frac{5\pi}{4} = \sin\left(\pi + \frac{\pi}{4}\right) = -\sin \frac{\pi}{4} = \frac{-\sqrt{2}}{2}; \\
 \tan \frac{5\pi}{4} &= \frac{\frac{-\sqrt{2}}{2}}{\frac{-\sqrt{2}}{2}} = 1.
 \end{aligned}$$

### ■ Examples B

Find the argument of the function that results in the value given.

$$[1] \quad \sin x = \frac{1}{2}.$$



Solution. Note that  $\sin \frac{\pi}{6} = \frac{1}{2}$ . Then,

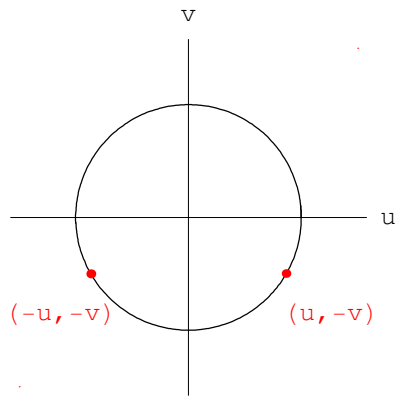
$$\sin \frac{\pi}{6} \implies x = \frac{\pi}{6}.$$

and

$$\sin \frac{\pi}{6} = \sin \left( \pi - \frac{\pi}{6} \right) \implies x = \pi - \frac{\pi}{6} \implies x = \frac{5\pi}{6}.$$

$$\therefore x = \frac{\pi}{6} + 2n\pi \text{ or } x = \frac{5\pi}{6} + 2n\pi, n \in \mathbb{Z}$$

$$[2] \sin x = \frac{-1}{2}.$$



Solution. Note that  $\sin \frac{\pi}{6} = \frac{1}{2}$ . Then,

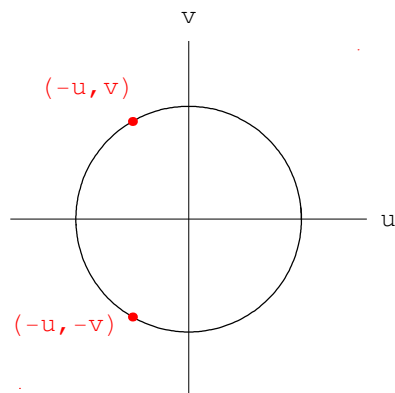
$$\frac{-1}{2} = -\sin \frac{\pi}{6} = \sin \frac{-\pi}{6} \implies x = \frac{-\pi}{6} = \frac{11\pi}{6} \implies x = \frac{11\pi}{6}.$$

and

$$\frac{-1}{2} = -\sin \frac{\pi}{6} = \sin \left( \pi + \frac{\pi}{6} \right) \implies x = \pi + \frac{\pi}{6} = \frac{7\pi}{6} \implies x = \frac{7\pi}{6}.$$

$$\therefore x = \frac{11\pi}{6} + 2n\pi \text{ or } x = \frac{7\pi}{6} + 2n\pi, n \in \mathbb{Z}.$$

$$[3] \cos x = \frac{-1}{2}.$$



Solution. Note that  $\cos \frac{\pi}{3} = \frac{1}{2}$ . Then,

$$\frac{-1}{2} = -\cos \frac{\pi}{3} = \cos \left( \pi - \frac{\pi}{3} \right) \implies x = \frac{2\pi}{3}.$$

and

$$\frac{-1}{2} = -\cos \frac{\pi}{3} = \cos \left( \pi + \frac{\pi}{3} \right) \implies x = \frac{4\pi}{3}.$$

$$\therefore x = \frac{2\pi}{3} + 2n\pi \text{ or } x = \frac{4\pi}{3} + 2n\pi, n \in \mathbb{Z}.$$

The following examples will not show the unit circle, but you should think of it.

[4]  $\cos x = \frac{-\sqrt{3}}{2}$ .

Solution. Note that  $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$ . Then,

$$\frac{-\sqrt{3}}{2} = -\cos \frac{\pi}{6} = \cos \left( \pi - \frac{\pi}{6} \right) \implies x = \frac{5\pi}{6}.$$

and

$$\frac{-\sqrt{3}}{2} = -\cos \frac{\pi}{6} = \cos \left( \pi + \frac{\pi}{6} \right) \implies x = \frac{7\pi}{6}.$$

$$\therefore x = \frac{5\pi}{6} + 2n\pi \text{ or } x = \frac{7\pi}{6} + 2n\pi, n \in \mathbb{Z}.$$