[06-12-19-T11]

Extension of exponents

■ Positive integer exponent

When n is a positive integer, a^n is easy to understand. We say

$$a^n = \underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{\text{n factors of } a}$$

When n is a negative integer, zero, a rational number, or an irrational number it is not so obvious what the symbol a^n means. Here, you will learn the definitions for such cases. Ideally, you will see why the definitions given are reasonable and useful.

■ a^0

Consider the rule $a^m \cdot a^n = a^{m+n}$ when $n = 0$.

$$a^m \cdot a^0 = a^{m+0} = a^m$$

Notice that a^0 behaves just as does 1, the multiplicative identity element.

◆ Definition a^0

For any real number a other than zero,

$$a^0 = 1$$

■ a^n, $n \in \{-1, -2, -3, \ldots\}$.

Let n be a positive integer, in which case $-n$ is a negative integer. Then,

$$a^n \cdot a^{-n} = a^{n+(-n)} = a^0 = 1.$$

a^{-n} behaves just as does $\frac{1}{a^n}$ the multiplicative inverse element of a^n.

◆ Definition a^n, $n \in \mathbb{Z}^-$

For any real number $a \neq 0$ and positive integer n (NB: if n is positive, $-n$ is negative),

$$a^{-n} = \frac{1}{a^n}$$

■ $a^{\frac{1}{n}}$, $n \in \{1, 2, 3, \ldots\}$

Suppose, for example, that $n = 2$. Then,

$$a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a.$$

$a^{\frac{1}{2}}$ behaves just as does $\sqrt{a}$ the square root of *a*.

◆ Definition $a^{\frac{1}{n}}$

a. If *n* is even positive integer and $a > 0$, then $a^{\frac{1}{n}} = \sqrt[n]{a}$

b. If *n* is odd positive integer, then $a^{\frac{1}{n}} = \sqrt[n]{a}$

c. If *n* is a positive integer, then $0^{\frac{1}{n}} = 0$

■ $a^{\frac{m}{n}}$, $m, n \in \{1, 2, 3, \ldots\}$

Since $a^{\frac{1}{n}} = \sqrt[n]{a}$ and $a^{\frac{m}{n}} = a^{\frac{1}{n} \cdot m}$, consider $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$

$$\left(\sqrt[n]{a}\right)^m = \underbrace{\sqrt[n]{a} \cdot \sqrt[n]{a} \cdot \sqrt[n]{a} \cdot \cdots \cdot \sqrt[n]{a}}_{\text{m factors of } \sqrt[n]{a}}$$

◆ Definition $a^{\frac{m}{n}}$

If *m, n* are positive integers, then $a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$, providing $\sqrt[n]{a}$ exists.

■ a^x, $x \in \mathbb{R}$

Defining a^x, $x \in \mathbb{R}$ depends on giving a definition of a^x when *x* is irrational. A completely satisfactory definition requires an understanding of real numbers that you have not yet acquired. We can say this much: you know how to approximate an irrational number by a rational number to any desired accuracy (e.g. newton's method). Perhaps you will accept, for the time being, the idea that when *b* is a positive real number and *x* is any real number, the symbol b^x names a unique real number. And, while you are in the mood to buy things, let's throw in that for any positive real number *b*, $b \neq 1$, and any positive real number *k*, there is a number *x* such that $k = b^x$.

■ Notes

[1] 0^0 is undefined.

[2] We have been careful to give definitions that are consistent with the familiar rules for exponents; hence, those rules hold for integer, rational, and real exponents.