[Solutions J10]

pps 54-56 Chapter Exercises

Exercise A

■ p. 54

[1.1] $\left(3\sqrt{2}+2\sqrt{3}+\sqrt{6}\right)\left(3\sqrt{2}-2\sqrt{3}-\sqrt{6}\right)=\mathbf{-12\sqrt{2}}$. REMARK: If you see and use the difference of squares here, this problem takes seconds to do.

[1.2] $\frac{2\sqrt{5}-3\sqrt{2}}{3\sqrt{5}-4\sqrt{2}}=\frac{\left(3\sqrt{2}-2\sqrt{5}\right)\left(\left(4\sqrt{2}+3\sqrt{5}\right)\right.}{\left(4\sqrt{2}-3\sqrt{5}\right)\left(4\sqrt{2}+3\sqrt{5}\right)}=\frac{\left(-6+\sqrt{10}\right)}{-13}=\mathbf{\frac{6-\sqrt{10}}{13}}$.

[1.3] $\frac{\sqrt{6}}{\sqrt{3}-\sqrt{2}}-\frac{4\sqrt{3}}{\sqrt{2}+\sqrt{6}}-\frac{3\sqrt{2}}{\sqrt{6}-\sqrt{3}}=\mathbf{0}$.

[1.4] $\sqrt{\frac{2}{6-\sqrt{35}}}=\sqrt{\frac{2\left(6+\sqrt{35}\right)}{\left(6-\sqrt{35}\right)\left(6+\sqrt{35}\right)}}=\sqrt{12+2\sqrt{35}}=\sqrt{7+2\sqrt{35}+5}=\mathbf{\sqrt{7}+\sqrt{5}}$.

[1.5] $\sqrt{\frac{\sqrt{2}+1}{\sqrt{2}-1}}-\sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}}=\mathbf{2}$.

[2.0]

$$\frac{\sqrt{p}+\sqrt{q}}{\sqrt{p}-\sqrt{q}}+\frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}}$$

$$=\frac{\left(\sqrt{p}+\sqrt{q}\right)^2}{\left(\sqrt{p}-\sqrt{q}\right)\left(\sqrt{p}+\sqrt{q}\right)}+\frac{\left(\sqrt{p}-\sqrt{q}\right)^2}{\left(\sqrt{p}-\sqrt{q}\right)\left(\sqrt{p}+\sqrt{q}\right)}$$

$$=\frac{\left(p+2\sqrt{q}\sqrt{p}+q\right)+\left(p-2\sqrt{q}\sqrt{p}+q\right)}{(p-q)}$$

$$=\frac{2(p+q)}{(p-q)}$$

Since the sum, difference, product, and quotient of rational numbers is a rational number, and p, q rational, $\frac{2(p+q)}{(p-q)}$ is a rational number also.

[3.1] {2}

[3.2] {1, 9}

[3.3] {1, 2, 4, 6, 8, 9}

[4.0] $(2yz + zx - 2xy) - 2(2xy - 3yz + 4zx) = \mathbf{-6xy + 8zy - 7xz}$.

■ p. 55

[5.1] $x^3 - x^2 y - 30 x y^2 = x(x - 6y)(x + 5y)$

[5.2] $p^3 - p^2 q - p q^2 + q^3 = (p - q)^2 (p + q)$

[5.3] $x^2 - 2(a - 1)x - 2a + 1 = -(2a - x - 1)(x + 1)$

[5.4] $x^2 + ax - 3bx - 3ab = -(3b - x)(a + x)$

[5.5] $x^2 y + y^2 z - y^3 - x^2 z = (x - y)(x + y)(y - z)$

[5.6] $x^2 + 3xy - 5x + 2y^2 - 7y + 6 = (x + y - 2)(x + 2y - 3)$

[5.7] $(x^2 + 2x)^2 - 2(x^2 + 2x) - 3 = (x - 1)(x + 1)^2 (x + 3)$

[5.8] $9x^4 - 34x^2 y^2 + 25y^4 = (3x - 5y)(x - y)(x + y)(3x + 5y)$

[6]
REMARK: recall that two polynomials are equal if and only if coefficients of their corresponding terms are equal. For example $3x^2 + \beta x - 2 = 3x^2 + 5x - 2$ if and only if $\beta = 5$.

Suppose $x^3 + ax^2 + x + 2 - a$ is divisible by $x^2 + 2x - 1$. This means $x^3 + ax^2 + x + 2 - a = (x^2 + 2x - 1) \cdot Q$ and that the degree of Q must be 1. So Q is the linear expression $Ax + B$. Thus, $x^3 + ax^2 + x + 2 - a = (x^2 + 2x - 1)(Ax + B)$. This equality requires that $x^3 = Ax^3$, so that $A = 1$. Similarly, $2Bx - Ax = x$. We already know $A = 1$, so

$2B - 1 = 1 \Longleftrightarrow B = 1$. We now know that

$x^3 + a x^2 + x + 2 - a = (x^2 + 2x - 1)(x + 1) = x^3 + 3x^2 + x - 1$.

Now, $x^3 + a x^2 + x + 2 - a = x^3 + 3x^2 + x - 1$ only if the coefficients of x^2 are equal. Therefore, $a = 3$.

[7] $\frac{a^{2n} - a^{-n}}{a^{2n} + a^{-n}} = \frac{3^2 - 3^{-1}}{3^2 + 3^{-1}} = \frac{13}{14}$

Exercise B

[1]

LHS $= \sqrt{3 + \sqrt{6}} + \sqrt{3 - \sqrt{6}}$, RHS $= \sqrt{6 + 2\sqrt{3}}$. Since LHS and RHS both positive, if

$\text{LHS}^2 = \text{RHS}^2$, then LHS = RHS.

$\text{LHS}^2 = \left(\sqrt{3 + \sqrt{6}} + \sqrt{3 - \sqrt{6}}\right)^2 = 6 + 2\sqrt{(3 - \sqrt{6})(3 + \sqrt{6})} = 6 + 2\sqrt{3}$.

$\text{RHS}^2 = \left(\sqrt{6 + 2\sqrt{3}}\right)^2 = 6 + 2\sqrt{3}$.

Therefore, $\sqrt{3 + \sqrt{6}} + \sqrt{3 - \sqrt{6}} = \sqrt{6 + 2\sqrt{3}}$.

[2]

$\frac{7}{\sqrt{9} - \sqrt{2}} = \sqrt{9} + \sqrt{2}$ and $\frac{1}{\sqrt{5} - 2} = \frac{1}{\sqrt{5} - \sqrt{4}} = \sqrt{5} + \sqrt{4}$.

Compare the squares of $\sqrt{9} + \sqrt{2}$ and $\sqrt{5} + \sqrt{4}$ which are both positive.

$$
\begin{aligned}
&\left(\sqrt{9} + \sqrt{2}\right)^2 - \left(\sqrt{5} + \sqrt{4}\right)^2 \\
&= 11 + 2\sqrt{18} - \left(9 + 2\sqrt{20}\right) \\
&= 11 + \sqrt{72} - \left(9 + \sqrt{80}\right) \\
&= 11 - 9 + \sqrt{72} - \sqrt{80} \\
&> 11 - 9 + \sqrt{72} - \sqrt{81} \\
&= 11 - 9 + \sqrt{72} - 9 \\
&= -7 + \sqrt{72} \\
&= -\sqrt{49} + \sqrt{72} \\
&> 0
\end{aligned}
$$

[3.1] 3 [3.2] $3\sqrt{13}$ [3.3] 36

[4] $\frac{40 + 17\sqrt{5}}{2 + \sqrt{5}} = \frac{(40 + 17\sqrt{5})(2 - \sqrt{5})}{(2 + \sqrt{5})(2 - \sqrt{5})} = 5 + 6\sqrt{5}$. If $5 + 6\sqrt{5} = a + b\sqrt{5}$, then $a = 5$, $b = 6$.

[5] REMARK. This is an excellent problem.

Suppose $\frac{x^3}{x+1} - P = \frac{a}{x+1}$. Now $P = A\,Q + R$, where P, Q, R polynomials in x. So,

$\frac{x^3}{x+1} - P = \frac{a}{x+1}$
$\Longleftrightarrow x^3 - P(x+1) = a$
$\Longleftrightarrow x^3 = P\,(x+1) + a.$

Then dividing x^3 by $(x+1)$ will make P and a obvious.

Doing so, reveals that $P = x^2 - x + 1$ and $a = -1$.

■ p. 56

[6.1] $4\,a^2 + 4\,b^2 + 4\,c^2$ [6.2] $\frac{2}{x-y}$ [6.3] 0

[7.1] $\sqrt{19 - 8\sqrt{3}} = \sqrt{19 - 2\sqrt{16 \cdot 3}} = \sqrt{3 - \sqrt{3}\,\sqrt{16} + 16} = \sqrt{\left(\sqrt{3} - \sqrt{16}\right)^2} = \sqrt{3} - 4$

[7.2]

Let $a \in \mathbb{Z}$, $b \in \mathbb{R}$, $0 \le b < 1$. Suppose $\sqrt{19 - 8\sqrt{3}} = a + b$, find $\frac{1}{b} - a$.

Obviously $b \neq 0$. Assume solution of [7.1]; that is, $\sqrt{19 - 8\sqrt{3}} = \sqrt{3} - 4$. Now a cannot be $\sqrt{3}$,

$a = -4$, $b = \sqrt{3}$. Then $\frac{1}{b} - a = \frac{1}{\sqrt{3}} + 4 = \frac{\sqrt{3}}{3} + 4 = \frac{12+\sqrt{3}}{4}$.

[8]

Find P and Q such that GCD$[P, Q] = x - 2$ and LCM$[P, Q] = 3\,x^3 + 8\,x^2 - 13\,x - 30$.

Since $(x-2)$ is a divisor of both P and Q, $P = R(x-2)$ and $Q = S(x-2)$. Since

$3\,x^3 + 8\,x^2 - 13\,x - 30$ is a multiple of P and Q, $3\,x^3 + 8\,x^2 - 13\,x - 30 = R\,S\,(x-2)^2$.

Dividing $3\,x^3 + 8\,x^2 - 13\,x - 30$ by $(x-2)^2$ yields, $3\,x + 20$.

So $3\,x^3 + 8\,x^2 - 13\,x - 30 = (x-2)\,(x-2)\,(3\,x + 20)$.

We conclude that $P = x - 2$ and $Q = (x-2)\,(3\,x + 20)$.