

INVERSE

To find an inverse function, we need only to interchange the elements of the ordered pairs of the bijection. To achieve this for a permutation we swap the rows then (usually) rearrange the order of the columns so the elements in the first row are in ascending order.

$$\text{For example, } \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$$

$$\text{as } \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = e$$

EXERCISE 9F.1

1 Simplify the following compositions of permutations:

$$\mathbf{a} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix} \quad \mathbf{b} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}$$

$$\mathbf{c} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \quad \mathbf{d} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

2 Find:

$$\mathbf{a} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}^{-1} \quad \mathbf{b} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}^{-1}$$

$$\mathbf{c} \left[\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} \right]^{-1}$$

3 Prove that, for all permutations p, q on $\{1, 2, 3, 4\}$, $(qp)^{-1} = p^{-1}q^{-1}$.

4 Find permutations p on $\{1, 2, 3, 4\}$ such that:

$$\mathbf{a} p \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} \quad \mathbf{b} p \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$$

5 For each of the following, construct a Cayley table and determine whether the set of permutations is a group under composition of permutations.

$$\mathbf{a} \{A, B, C, D\} \text{ where } A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

$$\mathbf{b} \{A, B, C, D\} \text{ where } A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

Is either **a** or **b** a cyclic group?

6 Explain why the group consisting of all the permutations on $\{1, 2, 3, 4, 5\}$ under composition of permutations has no subgroups of order 7.

EXERCISE 9E.4

- 1 **b** no
c $\{1\}$, $\{1, 5\}$, $\{1, 7\}$, $\{1, 11\}$, $\{1, 5, 7, 11\}$ under \times_{12} are subgroups
 3 **b** no **c** **i** a subgroup **ii** a subgroup

EXERCISE 9F.1

- 1 **a** $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$ **b** $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$
c $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$ **d** $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$
 2 **a** $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$ **b** $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$ **c** $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$
 4 **a** $p = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$ **b** $p = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}$

5 **a**

	A	B	C	D
A	A	B	C	D
B	B	C	D	A
C	C	D	A	B
D	D	A	B	C

closed, identity **A**,
inverse of **A** is **A**
of **B** is **D**
of **C** is **C**
of **D** is **B**

Is isomorphic to $\{0, 1, 2, 3\}$ under $+4$.

+4	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

a known Abelian
group
{Example 55}

$B^1 = B$, $B^2 = C$, $B^3 = CB = D$, $B^4 = DB = A$
 So, is a cyclic group.

b

	A	B	C	D
A	A	B	C	D
B	B	A	D	C
C	C	D	A	B
D	D	C	B	A

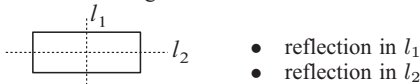
is isomorphic to

\times_8	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

which is a group
 So, is a group, but is not cyclic.
 $\{1, 3, 5, 7\}$ under \times_8 is easily checked to be non-cyclic.

EXERCISE 9F.2

- 2 • Anti-clockwise rotation about the centre of the rectangle through 0° .
 • Likewise but through 180° .



REVIEW SET 9A

- 1 **a** $\{a, b, c, d, e, f, g, h\}$ **b** $\{a, b, d, f\}$
c $\{a, b, d, f, g, h\}$
 2 $A \times B = \{(1, 2), (1, 4), (2, 2), (2, 4), (3, 2), (3, 4)\}$
 6 $\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$
a no **b** no
 7 **a** No, e.g., $(1 * 2) * 1 = 3 * 1 = \frac{4}{9}$
 whereas $1 * (2 * 1) = 1 * \frac{3}{4} = \frac{7}{4}$
 i.e., $(1 * 2) * 1 \neq 1 * (2 * 1)$

- b** No, e.g., $(0 * 1) * 2 = 2 * 2 = 2^4$
 $0 * (1 * 2) = 0 * 8 = 2^8$
 i.e., $(0 * 1) * 2 \neq 0 * (1 * 2)$
c Yes, $(a * b) * c = (a + b - 3ab) * c = a + b + c - 3ab - 3(a + b - 3ab)c = a + b + c - 3ab - 3ac - 3bc + 9abc \dots (1)$
 and likewise show $a * (b * c)$ is also equal to (1).
 8 **b** Each integer belongs to exactly one equivalence class containing all integers which have the same remainder on division by 6 as that integer. These are the six equivalence classes $[0], [1], [2], [3], [4]$ and $[5]$.
 9 **b** Each point (a, b) is an element of an equivalence class containing all points lying on a square, centre $(0, 0)$ with vertex at $(|a| + |b|, 0)$.
 $\{(0, 0)\}$ is an equivalence class with only 1 element.
 10 **b** Each point (a, b) belongs to an equivalence class consisting of all points on the parabola $y = \frac{b}{a^2}x$, excluding $(0, 0)$ where $\frac{b}{a^2} > 0$.
 11 Not correct as this does not show that $x R x$ for all $x \in S$. x may not be related to any other element in the set.
 12
- | | | | | | | |
|---|---|---|---|---|---|---|
| * | 0 | 1 | 2 | 3 | 4 | 5 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 2 | 3 | 4 | 5 | 0 |
| 2 | 4 | 0 | 2 | 4 | 0 | 2 |
| 3 | 3 | 0 | 3 | 0 | 3 | 0 |
| 4 | 4 | 2 | 0 | 4 | 2 | 0 |
| 5 | 1 | 0 | 5 | 4 | 3 | 2 |
- 13 **a** **i** no **ii** yes **iii** does not exist **iv** not possible
b **i** no **ii** no **iii** does not exist **iv** not possible
c **i** no **ii** yes **iii** does not exist **iv** not possible
d **i** no **ii** no **iii** does not exist **iv** not possible
e **i** yes **ii** yes **iii** 0 **iv** $-\frac{a}{1+3a}$, $a \neq -\frac{1}{3}$
f **i** no **ii** no **iii** does not exist **iv** not possible
 14 **a** yes, $f^{-1}(x) = \sqrt[3]{x-5}$ **b** yes, $f^{-1}(x) = e^x$
c no **d** yes, $f^{-1}(x) = \frac{1}{2}x$ **e** no
 15 **a** **i** $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$ **ii** $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$
b **i** $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$ **ii** $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$ **c** $n = 3$
 16 *Associativity* holds as 2×2 matrix multiplication is associative.
Closure holds as the product of any 2×2 matrix is always a 2×2 matrix.
 $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$ is the *identity* matrix, $a = 0$.
 The *inverse* of $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix}$ for all $a \in \mathbb{Z}$.
 $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$
 for all $a, b \in \mathbb{Z}$.
 \Rightarrow *commutativity* holds
 Thus, M under matrix multiplication forms an Abelian group.