If we investigate a very similar mathematical system consisting of a set $\{a, b, c\}$ and an operation $\circ$ defined by

$\circ$	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

we shall see that the closure, commutative, and associative properties are all satisfied.

Example. In the above system evaluate $(b \circ b) \circ c$ and $b \circ (b \circ c)$.

Solution: Using the table:

$$(b \circ b) = c$$
$$\therefore (b \circ b) \circ c = c \circ c$$
$$= b$$

$$(b \circ c) = a$$
$$\therefore b \circ (b \circ c) = b \circ a$$
$$= b$$

Exercises [A]

1. Is the set of natural numbers closed under multiplication?
2. Is the operation of multiplication on the natural numbers (a) commutative? (b) associative?
3. Is the set of natural numbers closed under subtraction?
4. Is the operation of subtraction on the natural numbers (a) commutative? (b) associative?
5. Consider the system consisting of $\{a, b, c\}$ and operation $\circ$ as defined above.
   (a) Read the values of $a \circ b$ and $b \circ a$.
   (b) Read the values of $c \circ b$ and $b \circ c$.
   (c) Read the values of $c \circ a$ and $a \circ c$.
   Is the operation $\circ$ on $\{a, b, c\}$ commutative?
6. For the system of Exercise 5, state clearly how you know that $\{a, b, c\}$ is closed under $\circ$.
7. For the system of Exercise 5, evaluate $(c \circ b) \circ b$ and $c \circ (b \circ b)$.
8. A system consists of the set $S = \{a, b\}$ and the operation $\odot$ defined by the table at the right.

$\odot$	a	b
a	a	b
b	b	a

   Determine whether the set is closed under $\odot$. Is the operation $\odot$ on S (a) commutative? (b) associative?

9. Show that $\{1, -1\}$ under ordinary multiplication is an example of the abstract system of Exercise 8.

10. Show that {even integer, odd integer} under ordinary addition is an example of the abstract system of Exercise 8.

11. $S = \{1, 2, 3, 4, 5, 6\}$. Let $\circ$ be defined by $x \circ y = x$ when $x, y \in S$. (a) Is S closed under $\circ$? (b) Is $\circ$ a commutative operation on S? (c) Is $\circ$ an associative operation on S?

12. Let $\square$ be defined on the set N of natural numbers by $x \square y =$ the greatest common factor of x and y. (a) Is N closed under the operation $\square$? (b) Is $\square$ a commutative operation on N? (c) Is $\square$ an associative operation on N?

13. Let $\circ$ be defined on the set N of natural numbers by $x \circ y = 1 + xy$. (a) Is N closed under the operation $\circ$? (b) Is $\circ$ commutative on N? (c) Is $\circ$ associative on N?

14. A mathematical system consists of $\{a, b, c, d\}$ and the operation $\otimes$ defined by the following table.

$\otimes$	a	b	c	d
a	d	c	b	a
b	c	a	d	b
c	b	d	a	c
d	a	b	c	d

(a) Is the set $\{a, b, c, d\}$ closed under $\otimes$?

(b) Is the operation $\otimes$ commutative? (Note that the elements in the table are arranged in a pattern which is symmetrical about the indicated diagonal. Does this indicate commutativity?)

(c) Evaluate: $(a \otimes b) \otimes c$ and $a \otimes (b \otimes c)$
$(b \otimes d) \otimes a$ and $b \otimes (d \otimes a)$
$(c \otimes b) \otimes d$ and $c \otimes (b \otimes d)$

15. Consider the set of all subsets of some universal set U.

(a) Is the set closed under the operation $\cup$ of set union?

(b) Is the operation commutative?

(c) Is the operation associative?

16. Consider the set of all subsets of some universal set U.

(a) Is the set closed under the operation $\cap$ of set intersection?

(b) Is the operation commutative?

(c) Is the operation associative?