Basic Ideas

DEFINITION OF BINARY OPERATION. An operation by which two elements of a set produce a single member of the set is called a *binary operation*.

DEFINITION OF CLOSED. An operation on a set is *closed* on a set if every possible combination under that operation is a member of the set.

By definition, a binary operation is closed. If an operation is not closed then it is not a binary operation.

EXAMPLE 1. Addition on the set of natural numbers is a binary operation, because the sum of two natural numbers is a natural number.

EXAMPLE 2. Subtraction on the set of natural numbers is not a binary operation, because the difference of two natural numbers is not always a natural number, for example $5 - 7 = -2$ and -2 is not a natural number.

DEFINITION OF COMMUTATIVE. A binary operation $\circ$ on a set S is *commutative* if for all elements $a, b \in S$, $a \circ b = b \circ a$.

EXAMPLE 3. Addition on the integers is commutative, because for any two integers a and b $a + b = b + a$; for instance $2 + 8 = 8 + 2$.

EXAMPLE 4. Subtraction on the integers is not commutative, because there is at least one pair of integers, for which $a + b \neq b + a$; for instance, $9 - 4 \neq 4 - 9$.

DEFINITION OF ASSOCIATATIVE. A binary operation $\circ$ on a set S is *associative* if for all elements $a, b, c \in S$, $a \circ (b \circ c) = (a \circ b) \circ c$.

EXAMPLE 5. Addition on the integers is associative. An illustration of this is $(2 + 3) + 8 = 2 + (3 + 8)$.

Consider the set $A = \{p, q, r\}$ and the binary operation $\oplus$ defined by the table

⊕	p	q	r
p	p	q	r
q	q	r	p
r	r	q	p

Is $\oplus$ a commutative operation on A? (You should be able to show that it is not.)

Is $\oplus$ an associative operation on A? (You should be able to show that it is not.)

On the other hand, you should be able to show that ○ is both commutative and associative on $A = \{a, b, c\}$ where ○ is defined by

○	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b