Math 11

Final Exam - Practice - ANSWERS

■ A.

[1] 4

[2] $\frac{1}{3}$

[3] $\sqrt{17}$

■ B.

[1] No solution

[2] $x = -5$

[3] $x = -2$ or $x = 1$

■ C.

[1] $\log_3 81 = 4$

[2] $2^4 = 16$

■ D.

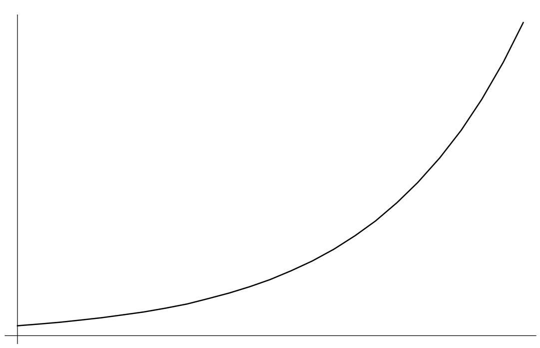
[1] 2

[2] 3.168

[3] 5

■ E.

[1]



The points $(0, 1)$, $(1, \pi)$ could be ploted and labeled.

■ F.

[1] $\text{DMN}_f = \mathbb{R} - \{-5, 3\}$

[2] $g(x) = 3x + 8,\ f(x) = \sqrt{x}$

■ G.

[1] $-\frac{\sqrt{3}}{2}$

[2] $-\frac{1}{2}$

[3] $-\frac{1}{\sqrt{3}}$

[4] $-\sqrt{2}$

[5] $\frac{\sqrt{6}-\sqrt{2}}{4}$

■ H.

[1] $x = \frac{\pi}{3}$ or $\frac{2\pi}{3}$

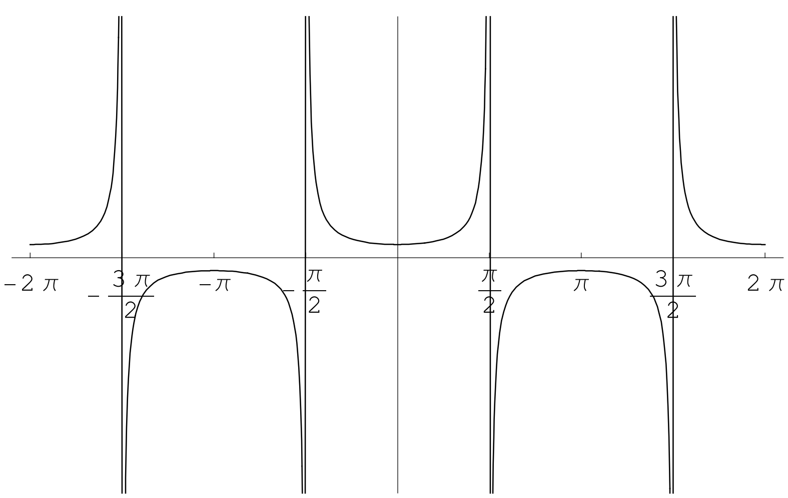
[2] $x = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$

[3] $x = \frac{\pi}{3}$ or $\frac{4\pi}{3}$

[4] $x \in \left[\frac{\pi}{3}, \frac{5\pi}{3}\right]$

■ I.

[1] Graph $y = \sec x$. Show at least one full period.



[2]

$f(x) = 2\sin 2x$

[3]

$f(x) = \sin 4\left(x + \frac{\pi}{4}\right)$

■ J.

[1] $\sin 3x = \sin 7x$

$\sin 3x = \sin 7x \Longrightarrow 3x = 7x$ or $3x = \pi - 7x$

$3x = 7x + 2n\pi \Longleftrightarrow 4x = 2n\pi$, so $x = \frac{2n\pi}{4} = \frac{n\pi}{2}$, $n \in \mathbb{Z}$

$3x = \pi - 7x + 2n\pi \Longleftrightarrow 10x = \pi - 2n\pi \Longleftrightarrow x = \frac{\pi}{10} - \frac{n\pi}{5}$, $n \in \mathbb{Z}$

$\boldsymbol{\therefore x \in \left\{\frac{2n\pi}{4},\ n \in \mathbb{Z}\right\} \cup \left\{\frac{\pi}{10} - \frac{n\pi}{5},\ n \in \mathbb{Z}\right\}}$

[2] $\sin 2\theta = \cos 3\theta$

$\cos 3\theta = \sin\left(\frac{\pi}{2} - 3\theta\right)$

$2\theta = \frac{\pi}{2} - 3\theta + 2n\pi$

$\theta = \frac{\pi}{10} + \frac{2n\pi}{5}$

OR

$2\theta = \pi - \left(\frac{\pi}{2} - 3\theta + 2n\pi\right)$

$2\theta = \pi - \frac{\pi}{2} + 3\theta + 2n\pi$

$\theta = -\frac{\pi}{2} + 2n\pi$

$\boldsymbol{\therefore x \in \left\{\frac{\pi}{10} + \frac{2n\pi}{5},\ n \in \mathbb{Z}\right\} \cup \left\{-\frac{\pi}{2} + 2n\pi,\ n \in \mathbb{Z}\right\}}$

■ K. Answer the following.

[1] In an arithmetic sequence, the 5^{th} term is *22* and the 10^{th} term is *47*. Find the 16^{th} term.

$a_{16} = 77$

[2] A geometric series has $t_2 = 6$, $t_5 = 48$. Find S_{10}, the sum of the first 10 terms.

$a_{10} = 1536$

■ L. Prove <u>ONE</u> of the following.

[1] $\dfrac{\sin(x-y)}{\sin(x+y)} = \dfrac{\tan x - \tan y}{\tan x + \tan y}$

LHS

$= \dfrac{\sin x \cos y - \sin y \cos x}{\sin x \cos y + \sin y \cos x}$

$= \dfrac{\sin x \cos y - \sin y \cos x}{\sin x \cos y + \sin y \cos x}\left(\dfrac{\frac{1}{\cos x \cos y}}{\frac{1}{\cos x \cos y}}\right)$

$= \dfrac{\frac{\sin x \cos y}{\cos x \cos y} - \frac{\sin y \cos x}{\cos x \cos y}}{\frac{\sin x \cos y}{\cos x \cos y} + \frac{\sin y \cos x}{\cos x \cos y}}$

$= \dfrac{\tan x - \tan y}{\tan x + \tan y}$

$=$ RHS

[2] $1 + \cos^2 x = \dfrac{\tan^2 x + 2}{\sec^2 x}$

RHS

$= \dfrac{\tan^2 x + 2}{\sec^2 x}$

$= \dfrac{\frac{\sin^2 x}{\cos^2 x} + 2}{\frac{1}{\cos^2 x}}$

$= \dfrac{\frac{\sin^2 x + 2\cos^2 x}{\cos^2 x}}{\frac{1}{\cos^2 x}}$

$= \sin^2 x + 2\cos^2 x$

$= \sin^2 x + \cos^2 x + \cos^2 x$

$= 1 + \cos^2 x$

$=$ LHS

■ M. Answer the following. (10 points each)

[1] $(1 + i) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

[2] Use de Moivre's theorem to compute

$$
\begin{aligned}
&(1 + i)^6 \\
&= \left(\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)^6 \\
&= \left(\sqrt{2}\right)^6\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) \\
&= (8)(0 - i) \\
&= 8\,i
\end{aligned}
$$

.

[3] Use mathematical induction to prove that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Prove: $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$ for $n = 2, 3, 4, \ldots$.

Proof.

Let S_n be the statement $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$, $n = 1, 2, 3, \ldots$.

Basis. Verify that S_1 is true. S_1 is $1 = \frac{1\,(1+1)}{2}$. Clearly true.

Inductive step. Show that *if* S_k is true *then* S_{k+1} must be true, $k \in \mathbb{Z}^+$.

Suppose S_k is true for some $k \in \mathbb{Z}^+$, i.e.

S_k: $1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}$. [Note: statement S_k is known as the induction hypothesis.]

Show that then S_{k+1} must be true, i.e.

S_{k+1}: $1 + 2 + 3 + \ldots + (k + 1) = \frac{(k+1)\,((k+1)+1)}{2} = \frac{(k+1)\,(k+2)}{2}$.

LHS $= 1 + 2 + 3 + \ldots + k + (k + 1)$

$= \frac{k(k+1)}{2} + (k + 1)$ [Note: here we have used the hypothesis, statement S_k.]

$= \frac{k\,(k+1)+2\,(k+1)}{2}$

$= \frac{(k+1)\,(k+2)}{2}$

= RHS

Conclusion. If S_k is true, then S_{k+1} is true. Since S_1 is true, S_1, S_2, S_3, … are all true.

$\therefore$ S_n is true. $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$, $n \in \mathbb{Z}^+$

□