■ The point-slope equation of a straight line

We wish to derive an equation for the straight line ℓ.

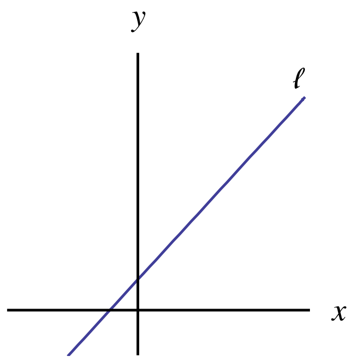
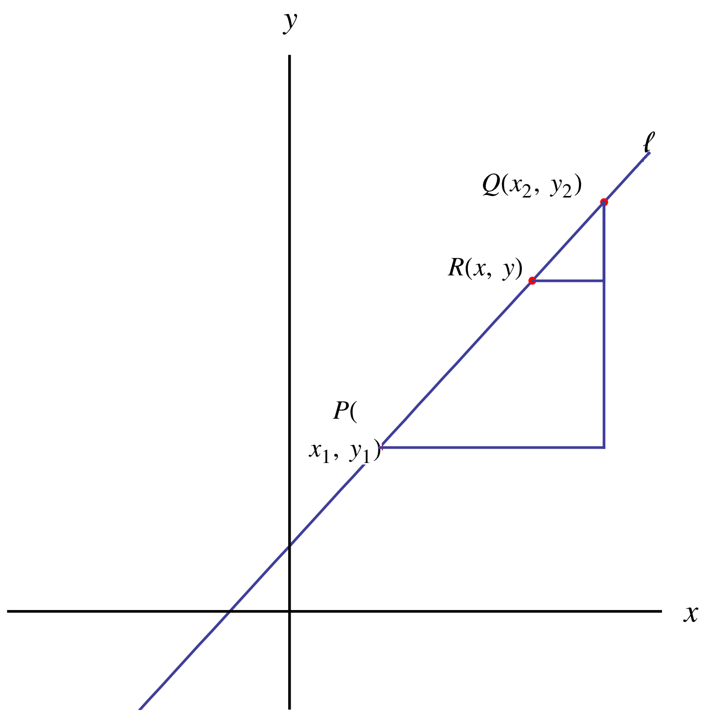


Figure 1

Before we do so, we must characterize a straight line. We might say that a line is "straight" when its direction is unchanging. Another way to put this is to say that a line is straight when its slope is constant. But, what do we mean by the phrase "its slope is constant"? This should work: the slope of a line is constant when it is the same no matter which two points on the line we use to compute it. We shall use this characterization of a straight line in the derivation below.

Choose *any* two fixed points on the line ℓ. Call them points P and Q. Call the first coordinate of point P x_1 and the second coordinate of point P y_1. Call point Q's first and second coordinates x_2 and y_2 respectively. So, the two fixed points on the line ℓ are $P(x_1,\ y_1)$ and $Q(x_2,\ y_2)$.

Next choose a point that is on line but allowed to move freely along the line ℓ. Call this point R and its coordinates x and y.



Now, let us compute the slope of the line ℓ using points P and Q. We will label this slope m_{PQ} meaning "the slope of line ℓ using points P and Q". Thus,

$$m_{PQ} = \frac{y_2 - y_1}{x_2 - x_1}. \tag{1}$$

Next, compute the slope of the line ℓ using points P and R. We will label this slope m_{PR} meaning "the slope of line ℓ using points P and R". Thus,

$$m_{PR} = \frac{y - y_1}{x - x_1}. \tag{2}$$

Now we use the fact that the line ℓ is a straight line. Since ℓ is a straight line, its slope must be the same no matter which two points on the line we use to compute the slope. But, this means that

$$m_{PR} = m_{PQ} \tag{3}$$

Using equations (1) and (2), substitute $\frac{y_2 - y_1}{x_2 - x_1}$ for m_{PQ} and $\frac{y - y_1}{x - x_1}$ for m_{PR} in equation (3). Thus,

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}. \tag{4}$$

Multiply both sides of equation (4) by $x - x_1$ to get

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1). \tag{5}$$

Since the slope of line ℓ is the same regardless of which points we use to compute it, we might as well just call that slope m. Using the name m for the slope of line ℓ in equation (5), we write

$$\boldsymbol{y - y_1 = m(x - x_1)}. \tag{6}$$

Equation (6) is known as the point-slope equation of the straight line.

Here's what we accomplished. We now know that any straight line must have an equation of the form $y - y_1 = m(x - x_1)$ where x_1 and y_1 are the coordinates of any point on the line and where m is the slope of the straight line.

□

■ Example 1

Find the equation of the line through the points (2, 3) and (7, 13).

Solution.

The equation of this line must be of the form $\boldsymbol{y - y_1 = m(x - x_1)}$, because *every* straight line has an equation of this form. Once x_1 and y_1 are replaced by the coordinates of a point on the line and the slope m is computed, we need merely insert them into $\boldsymbol{y - y_1 = m(x - x_1)}$ and we are done.

Since both the points (2, 3) and (7, 13) are on the line, the coordinates of either point will do for x_1 and y_1.

The slope m is easily found because we know two points on the line, $m = \frac{13-3}{7-2} = \frac{10}{5} = 2$.

Therefore, the equation of the line through the points (2, 3) and (7, 13) is $\boldsymbol{y - 3 = 2\,(x - 2)}$. □

■ Example 2

Find the equation of the line through the points (−2, 3) and (7, −13).

Solution.

Reasoning as in example 1, we write $m = \frac{-13-3}{7+2} = \frac{-16}{9}$. So, the equation of the line through the points (−2, 3) and (7, −13) is $y - 3 = \frac{-16}{9}\,(x + 2)$. □

■ Example 3

Find the equation of the line through the point (2, −7) and whose slope is $\frac{-1}{2}$.

Solution.

This is too easy! The answer is $y + 7 = \frac{-1}{2}\,(x - 2)$. □

■ Example 4

Find the equation of the line ℓ_1 through the point $(2, -7)$ that is parallel to the line ℓ_2: $2x + 3y = 5$.

Solution.

The slope of ℓ_1 must be the same as the slope of ℓ_2. So, the task is to find the slope m_2 of ℓ_2. Two points on ℓ_2 are $\left(0, \frac{5}{3}\right)$ and $\left(\frac{5}{2}, 0\right)$. So $m_2 = \frac{0-\frac{5}{3}}{\frac{5}{2}-0} = \frac{-\frac{5}{3}}{\frac{5}{2}} = \frac{-5}{3} \cdot \frac{2}{5} = \frac{-2}{3}$. So, the equation of line ℓ_1 is $\boldsymbol{y + 7 = \frac{-2}{3}(x - 2)}$. □

■ Example 5

Find the equation of the line ℓ_1 through the point $(2, -7)$ that is perpendicular to the line ℓ_2: $2x + 3y = 5$.

Solution.

You learned that the product of the slopes of perpendicular lines is equal to -1. From the previous example, the slope of line ℓ_2 is $\frac{-2}{3}$. Thus, $m_2 m_1 = -1 \Longleftrightarrow \frac{-2}{3} m_1 = -1 \Longleftrightarrow m_1 = \frac{3}{2}$. Then the equation of the line through $(2, -7)$ and perpendicular to the line $2x + 3y = 5$ is $y + 7 = \frac{3}{2}(x - 2)$.□

Formulae

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Out[125]=